BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

EDITED BY

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Part I

VOLUME 47

Published by the
AMERICAN MATHEMATICAL SOCIETY
Providence, Rhode Island 02904

Reprinted 1969

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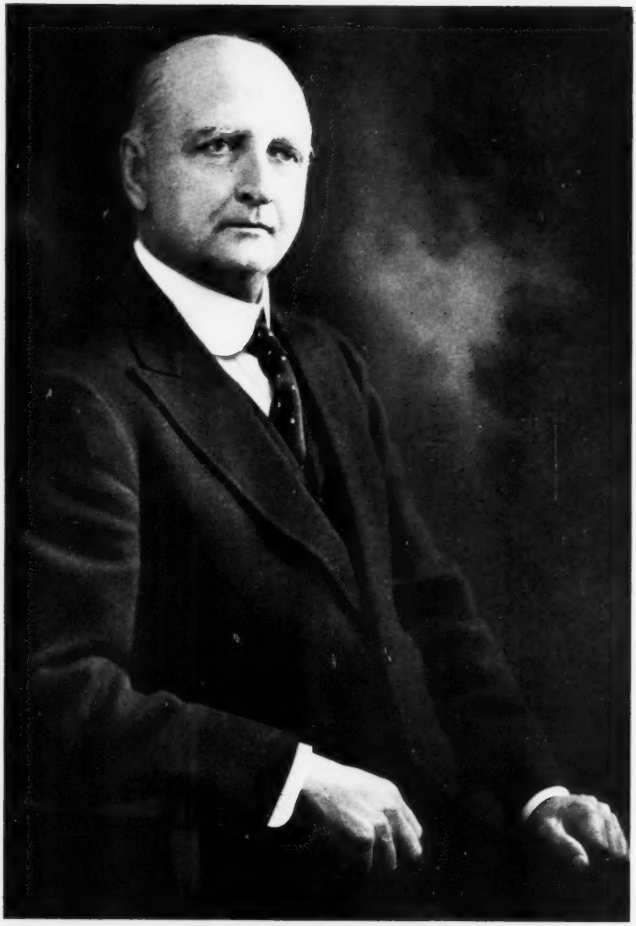

R. G. D. Richardson

ROLAND GEORGE DWIGHT RICHARDSON

By order of the Council of the Society,
this volume is dedicated to

ROLAND GEORGE DWIGHT RICHARDSON

in appreciation of his devoted service to the
Society especially during his term of
office as Secretary from
1921 to 1940

BULLETIN
OF THE
AMERICAN MATHEMATICAL SOCIETY

MATHEMATICAL REVIEWS OFFERS A READING MACHINE FOR MICROFILM

When the microfilm service of *Mathematical Reviews* was introduced, it was realized that its usefulness would depend to a large extent upon the availability of reading machines. The Committee on Scientific Aids to Learning, a committee of National Research Council, is promoting, among other things, the use of microfilm. As a result of its efforts, a reading machine is being manufactured which will be sold at a retail price of $32.00. A grant from the Committee on Scientific Aids to Learning has made it possible for *Mathematical Reviews* to distribute a limited number of these machines on the following terms.

Terms of offer. A reading machine for microfilm will be given—as long as the available supply lasts—to any person who has paid his subscription, *at the rate to which he is entitled*, to *Mathematical Reviews* in advance for three years beginning January, 1941. The person who receives a reading machine must pay express charges and import duty, if any, from Buffalo, New York. Until January 1 this offer was made only to the present subscribers to *Mathematical Reviews*. Since that date, however, it is extended to new subscribers also. Because only a limited number of machines is available, anyone who desires one should place an order early.

The purpose of the Committee on Scientific Aids to Learning in affording the financial support making possible the distribution of the readers in connection with subscriptions to *Mathematical Reviews* was jointly the promotion of microfilm and aid to *Mathematical Reviews*.

History of Students Microfilm Reader. The history of the reader, known as the Students Microfilm Reader, begins in the fall of 1939. At that time an advisory group on microphotography to the Committee on Scientific Aids to Learning, composed of Mr. Keyes D. Metcalf (chairman), Director of the Harvard University Library, Professors Ralph D. Bennett and Ernest I. Huntress of Massachusetts Institute of Technology, Dr. Vernon D. Tate of the National Archives, and Dr. Irvin Stewart (ex officio), Director of the Com-

mittee on Scientific Aids to Learning, was requested to consider the possibilities of designing and making available a simple, inexpensive microfilm reading machine for the use of the individual scholar. The problem was discussed at length and did not appear insoluble. Several designs were suggested, and three models were constructed. Each of these models was thoroughly tested both in the laboratory and in actual use; a set of plans and specifications embodying the final accepted design was prepared for distribution to manufacturers specializing in equipment of this type.

Bids for the manufacture of the reading machine were received from a number of companies, and the Spencer Lens Company was authorized to build a pilot model. It was built, tested and inspected, and the Committee on Scientific Aids to Learning has now signed a contract for a number of these machines. In addition, they will be placed on the market by the Spencer Lens Company.

Purpose of Students Microfilm Reader. Emphasis has been placed throughout on suitability for the purpose for which the reader was designed: simplicity and low cost. No claims are made for extreme convenience, beauty, ready-portability, or universality.

The benefits of microphotography in assembling research data of all types are well known. Facilities exist in the principal libraries, archives, and other institutions for the reproduction of their holdings. Many individuals have secured equipment and microphotographed extensive files. The greatest difficulty to date has not been to secure material on microfilm but rather to obtain adequate utilization equipment. In sponsoring the development of a simple, inexpensive microfilm reader, the Committee on Scientific Aids to Learning has considered solely the requirements of the individual. Excellent equipment developed primarily for commercial and library use is already available on the market. In most cases, it is entirely satisfactory (except for price) for individual use. The Students Microfilm Reader is not intended to compete with any existing reader equipment. It was developed specifically to permit the individual scholar or scientist to utilize in his own study or laboratory microphotographic copies which he may have made personally or procured from one of the existing sources of supply.

G. B. PRICE

THE OCTOBER MEETING IN NEW YORK

The three hundred seventy-second meeting of the American Mathematical Society was held at Columbia University on Saturday, October 26, 1940. The attendance included the following one hundred sixty-eight members of the Society:

Leon Alaoglu, C. B. Allendoerfer, Warren Ambrose, R. G. Archibald, K. J. Arrow, M. F. Becker, E. E. Betz, Gertrude Blanch, R. P. Boas, H. F. Bohnenblust, W. M. Bond, Samuel Borofsky, D. G. Bourgin, C. B. Boyer, A. T. Brauer, H. C. Brodie, A. A. F. Brown, A. B. Brown, Jewell H. Bushey, J. H. Bushey, S. S. Cairns, W. D. Cairns, J. W. Calkin, A. D. Campbell, E. W. Cannon, Achille Capecelatro, J. A. Clarkson, T. F. Cope, Byron Cosby, Richard Courant, H. B. Curry, M. D. Darkow, Norman Davids, M. R. Demers, C. H. Dix, Jesse Douglas, J. E. Eaton, M. L. Elveback, H. T. Engstrom, Paul Erdös, C. J. Everett, G. M. Ewing, J. M. Feld, Aaron Fialkow, E. J. Finan, G. E. Forsythe, R. M. Foster, J. S. Frame, K. O. Friedrichs, Orrin Frink, T. C. Fry, G. N. Garrison, H. P. Geiringer, B. P. Gill, J. W. Givens, Saul Gorn, S. L. Greitzer, H. M. Griffin, D. W. Hall, F. C. Hall, N. A. Hall, Hans-Karl Hammer, D. C. Harkin, Philip Hartman, M. H. Heins, Edward Helly, J. G. Herriot, E. H. C. Hildebrandt, Einar Hille, Banesh Hoffmann, Lulu Hofmann, T. R. Hollcroft, E. A. Hoy, R. Y. Hoy, E. M. Hull, Seymour Jablon, R. L. Jeffrey, S. A. Joffe, R. A. Johnson, Edward Kasner, L. S. Kennison, J. R. Kline, R. F. Koch, B. O. Koopman, Arthur Korn, A. W. Landers, M. K. Landers, Solomon Lefschetz, C. A. Lester, Howard Levi, Madeline Levin, Marie Litzinger, L. H. Loomis, E. R. Lorch, A. N. Lowan, Brockway McMillan, C. C. MacDuffee, H. F. MacNeish, Dorothy Maharam, P. T. Maker, A. J. Maria, D. H. Maria, Walther Mayer, A. E. Meder, H. L. Mintzer, Don Mittleman, E. C. Molina, Deane Montgomery, Marston Morse, Z. I. Mosesson, C. A. Nelson, Philip Newman, Oystein Ore, J. C. Oxtoby, Gordon Pall, W. F. Penney, B. J. Pettis, E. L. Post, Walter Prenowitz, C. J. Rees, M. S. Rees, H. J. Riblet, R. G. D. Richardson, J. F. Ritt, I. F. Ritter, H. E. Robbins, S. L. Robinson, Benjamin Rosenbaum, J. E. Rosenthal, Arthur Sard, Abraham Schwartz, I. E. Segal, Hyman Serbin, Stephan Serghiesco, L. W. Sheridan, Seymour Sherman, Max Shiffman, James Singer, Abraham Sinkov, L. L. Smail, P. A. Smith, Abraham Spitzbart, Wolfgang Sternberg, C. N. Stokes, M. H. Stone, W. C. Strodt, Alvin Sugar, M. M. Sullivan, W. G. Swope, J. D. Tamarkin, J. M. Thomas, Arthur Tilley, E. W. Titt, C. B. Tompkins, W. J. Trjitzinsky, J. W. Tukey, Annita Tuller, G. C. Webber, Louis Weisner, M. E. Wells, William Wernick, A. P. Wheeler, O. L. Wheeler, A. L. Whiteman, H. P. Wirth, Jack Wolfe, Leo Zippin, Antoni Zygmund.

The meeting opened Saturday morning with two sessions: Analysis, Professors Einar Hille and J. R. Kline presiding; and Algebra and Geometry, Professor S. S. Cairns presiding. At the afternoon session, at which Vice President T. C. Fry presided, Professor H. B. Curry gave an address entitled *Some aspects of the problem of mathematical rigor*.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t*

were read by title. The papers numbered 1 to 8 were read before the section for Analysis; those numbered 9 to 13 before the section for Algebra and Geometry; and those numbered 14 to 21 were read by title.

1. G. E. Forsythe: *On Riesz summability methods of order r, for* $\Re(r) < 0$. Preliminary report. (Abstract 46-11-488.)

2. Jenny E. Rosenthal: *Generating functions and properties of certain orthogonal polynomials.* II. (Abstract 46-11-501.)

3. I. E. Segal: *The space of Besicovitch almost periodic functions.* (Abstract 46-11-505.)

4. A. B. Brown: *On the number of independent parameters.* (Abstract 46-11-485.)

5. P. T. Maker: *A topological characterization of monotone functions.* Preliminary report. (Abstract 46-11-496.)

6. E. R. Lorch: *The integral representation of weakly almost-periodic transformations in reflexive vector spaces.* (Abstract 46-7-361.)

7. Einar Hille: *A class of differential operators of infinite order.* II. (Abstract 47-1-31.)

8. L. H. Loomis: *The decomposition of meromorphic functions into rational functions of univalent functions.* (Abstract 46-11-495.)

9. Saul Gorn: *Homomorphisms and modular functionals.* (Abstract 46-11-489.)

10. N. A. Hall: *The solution of the quadrinomial equation in infinite series by the method of iteration.* (Abstract 46-11-490.)

11. A. T. Brauer: *On a problem of partitions.* (Abstract 46-11-484.)

12. Aaron Fialkow: *The conformal theory of a hypersurface.* (Abstract 46-11-487.)

13. William Wernick: *Distributive properties of set operators.* (Abstract 46-11-506.)

14. R. S. Martin: *Minimal positive harmonic functions.* (Abstract 47-1-37-*t*.)

15. Francisco Perez: *A generalization of the theory of invariant factors and similar matrices.* (Abstract 46-11-498-*t*.)

16. L. B. Robinson: *A functional equation with negative exponent.* (Abstract 46-11-500-*t*.)

17. L. B. Robinson: *A functional equation with a singular line.* (Abstract 46-11-499-*t*.)

18. Peter Scherk: *On real closed curves of the order* $n+1$ *in the projective n-space.* Preliminary report. (Abstract 46-11-502-*t*.)

19. H. S. Wall: *Real power series bounded in the unit circle.* (Abstract 46-9-440-*t*.)

20. William Wernick: *Functional dependence in the calculus of propositions.* (Abstract 46-11-507-*t*.)

21. Hassler Whitney: *The mappings of a 3-complex into a space with vanishing fundamental group.* (Abstract 46-11-508-*t*.)

T. R. HOLLCROFT,
Associate Secretary

THE NOVEMBER MEETING IN DETROIT

The three hundred seventy-third meeting of the American Mathematical Society was held at Wayne University, Detroit, Michigan, on Friday and Saturday, November 22–23, 1940. Ninety-five persons registered including the following sixty-nine members of the Society:

Max Astrachan, J. V. Atanasoff, W. L. Ayres, I. A. Barnett, E. F. Beckenbach, R. A. Beth, W. M. Borgman, H. K. Brown, R. S. Burington, R. V. Churchill, C. J. Coe, A. H. Copeland, Max Coral, C. C. Craig, J. M. Dobbie, M. J. Dunford, Ben Dushnik, W. D. Duthie, P. S. Dwyer, Samuel Eilenberg, A. G. Fleiger, K. W. Folley, Morris Friedman, H. H. Goldstine, V. G. Grove, O. G. Harrold, B. A. Hausmann, G. E. Hay, E. D. Hellinger, T. H. Hildebrandt, J. D. Hill, C. C. Hurd, F. B. Jones, Samuel Kaplan, Wilfred Kaplan, D. K. Kazarinoff, J. L. Kelley, E. D. McCarthy, A. V. Martin, D. D. Miller, E. W. Miller, D. C. Morrow, S. B. Myers, A. L. Nelson, C. J. Nesbitt, J. W. Odle, E. W. Paxson, E. C. Pixley, V. C. Poor, J. E. Powell, G. Y. Rainich, E. D. Rainville, C. E. Rickart, Arthur Rosenthal, J. B. Rosser, W. T. Scott, H. A. Simmons, B. M. Stewart, Otto Szász, Leonard Tornheim, A. W. Tucker, W. A. Vezeau, H. S. Wall, I. T. Weinberg, E. T. Welmers, K. P. Williams, L. A. Wolf, M. C. Wolf, J. W Zimmer.

The meeting opened Friday afternoon with a session for short papers, followed by an address by Professor J. B. Rosser of Cornell University entitled *Many-valued logics*. On Saturday morning Professor H. S. Wall of Northwestern University delivered an address under the title *Some recent developments in the theory of continued fractions*, and the meeting closed with a second session for short papers. Professor A. L. Nelson presided on Friday afternoon and Professor K. P. Williams on Saturday morning.

On Friday evening a dinner was held at the Belcrest Hotel with sixty-six persons attending. Professor A. L. Nelson presided at the dinner and Dean W. W. Whitehouse of Wayne University welcomed the Society in behalf of the institution. At the Saturday session Professor E. F. Beckenbach presented a resolution of appreciation to Wayne University and its Department of Mathematics for their hospitality.

The titles of papers read at this meeting follow, papers 1–6 being read Friday afternoon and 7–10 Saturday morning. Papers 11–13, whose abstract numbers are followed by the letter *t*, were read by title. Paper 2 was read by Professor Miller, 3 by Professor Wall, 4 by Dr. Scott.

1. K. W. Folley: *A property of a simply ordered set.* (Abstract 47-1-23.)

2. Ben Dushnik and E. W. Miller: *On the dimension of a partial order.* (Abstract 47-1-89.)

3. W. T. Scott and H. S. Wall: *Linear manifolds of Hausdorff means.* (Abstract 46-11-504.)

4. W. T. Scott and H. S. Wall: *A geometrical method in the theory of continued fractions.* (Abstract 46-11-503.)

5. G. Y. Rainich: *Postulates for fields.* Preliminary report. (Abstract 47-1-7.)

6. A. W. Tucker: *Barycentric mappings.* (Abstract 47-1-110.)

7. F. B. Jones: *Totally discontinuous linear functions whose graphs are connected.* (Abstract 47-1-93.)

8. J. W. Odle: *Non-separating and non-alternating transformations modulo a family of sets.* (Abstract 47-1-102.)

9. Samuel Eilenberg: *Linear measure and convexity.* (Abstract 47-1-22.)

10. H. A. Simmons: *Maximum numbers associated with a symmetric Diophantine equation in n reciprocals.* (Abstract 47-1-86.)

11. A. N. Milgram: *Extensible and inextensible decompositions.* Preliminary report. (Abstract 47-1-96-*t*.)

12. M. M. Day: *Reflexive Banach spaces which cannot be made uniformly convex.* (Abstract 46-9-396-*t*.)

13. M. M. Day: *Some more uniformly convex spaces.* (Abstract 46-11-486-*t*.)

W. L. AYRES,
Associate Secretary

BOOK REVIEWS

Advances and Applications of Mathematical Biology. By N. Rashevsky. Chicago, University Press, 1940. 18+214 pp. $2.00.

As the title indicates, the book under discussion deals with the more recent progress made by Rashevsky and his group in the systematization of what they have termed physico-mathematical biology. The problems with which the author is primarily concerned are those already delineated in his fundamental work published about two years ago, viz.: (a) cellular metabolism and growth, (b) conduction of nerve impulses and (c) neuropsychological reactions. In the present volume one finds that the analytical method first employed has been somewhat simplified and additional examples are given of the agreement between actual observations and some of the conclusions derived from mathematical reasoning. In the case of cellular diffusion, for example, the author has succeeded in developing formulations which are not restricted by the postulate that the cells are spherical. The elimination of this restriction obviously opens the way to a more general application of the author's rationalizations. Thus, he is enabled to outline with greater precision his concepts regarding the effects of diffusion forces on cell division.

This and the preceding book will probably interest the mathematician mainly because they reveal the kinds of important and vital problems that can be tackled with rather simple analytical tools. The biologist will undoubtedly be stimulated by the fresh ideas emerging from some of the mathematical developments, even though the unverifiable nature of some of the postulates will for the present hinder the experimental or observational evaluation of all the deductions that can be reached.

ANTONIO CIOCCO

Sur les Valeurs Exceptionnelles des Fonctions Méromorphes et de Leurs Dérivées. By Georges Valiron. (Actualités Scientifiques et Industrielles, no. 570.) Paris, Hermann, 1937. 55 pp.

One of the outstanding methods of the present century in the theory of functions of a complex variable is that of normal families of functions, a method which has well known applications to the study of convergence of sequences of analytic functions, the distribution of functional values of functions analytic or meromorphic in a circle or in the entire finite plane, conformal mapping, the theorems of Picard, Landau, Schottky, and so on. Each new condition for normal-

ity of a sequence of functions has yielded new information on one or more of these topics.

At least two decades ago, Montel conjectured that a family of functions $f(z)$ analytic in a region is normal there provided $f(z)$ fails to assume the value zero and $f'(z)$ fails to assume the value unity. This conjecture was discussed at least orally by many mathematicians, and was finally proved in 1935 by Miranda. Miranda had been preceded by Bureau, who obtained some related theorems by the use of the Nevanlinna theory of meromorphic functions but who did not establish Montel's conjecture. Miranda's work was based also on the Nevanlinna theory and on Bureau's results. Valiron subsequently introduced the direct methods that he had previously employed in the study of the theorems of Picard and Borel to obtain wide generalizations of Miranda's theorem.

The fascicle before us is devoted to a systematic exposition of this body of material, together with many new generalizations and complements. There are close connections with the theorems of Picard, Landau, Schottky, and Bloch. The proofs are far more than pure existence proofs, for they involve specific inequalities of a numerical nature on the modulus of the functions $f(z)$.

The treatment is clear, pleasing, suggestive—an admirable exposition of a field of current interest and importance. May there soon be made in this country systematic provision for the encouragement of the writing of similar essays and for their publication!

J. L. WALSH

Complex Variable and Operational Calculus with Technical Applications. By N. W. McLachlan. Cambridge, University Press, 1939. 11+355 pp. $6.50.

Since the time of Heaviside several books on operational calculus have appeared on the market. The majority of them deal almost exclusively with the electrical applications. This is due, no doubt, to its origin in the pioneer work of Heaviside. With very few exceptions these books treat the subject in a formal and heuristic manner leaving much to be desired from the standpoint of mathematical rigor. Several mathematicians have helped to place the subject on a more substantial basis and thereby extend its usefulness. The author has based his treatment of the subject on the Laplace transform and the Mellin inversion theorem in line with modern developments. Except in a few places a sufficient degree of rigor has been maintained throughout the book to meet the needs of the engineer and applied mathematician.

The book is divided into three parts and an appendix. Part I is a treatment of sections of complex variable theory with a view to applications. The second part deals with the operational calculus. In the third part we find the applications to many types of problems encountered in practice. This is followed by an appendix dealing with some theoretical questions and an extensive bibliography.

The first chapter plunges the reader into the theory of complex variables. No time is wasted before the introduction of singular points. An essential singularity is defined as a pole of infinite order which is rather loose. The principal part of the Laurent expansion provides a clear as well as a rigorous definition of the singularities of a single valued function, but this is postponed until the second chapter. This is an example where suggestive language does not lead to clarity but where rigor does not imply a lack of vigor. A few slight lapses like this one mar an otherwise admirable presentation. Branch points receive a very good treatment from the point of view of their use in the applications. The Cauchy-Riemann test for analyticity is completely left out. No doubt, all the functions arising in applications are analytic, but the reader ought to be provided with this important criterion. Integration is taken up in the second chapter. Cauchy's theorem is stated without proof. The Cauchy integral formula, Taylor's theorem and the Laurent expansion round out the chapter. Integrals indefinitely close together and infinitesimal gaps in the contours provide another example of loose but suggestive language.

A very thorough but not always rigorous treatment of the calculus of residues and contour integration around branch points is given in chapter three. This leads up to the treatment of the Bromwich-Wagner integral in the next chapter. By the use of Jordan's lemma and deformation of the contour several important cases of the Bromwich integral receive a clear and thorough treatment. The author closes the fourth chapter with some very neat methods to obtain approximate values of integrals needed in the sequel.

Three of the most useful functions are the Gamma, Bessel, and Error functions. In the sixth chapter an adequate treatment of these functions is provided as well as a discussion of asymptotic expansions and analytic continuation. When branch points occur the treatment of the Bromwich integral is very fussy and the author has treated it with some care. A short chapter on differentiation under the integral sign and other iterated limit problems closes the first part of the book.

Part II introduces the operational calculus on the basis of the p-multiplied form of the Laplace transform and Mellin inversion

theorem. The conditions under which the inversion is possible are narrow but suitable enough for the purpose at hand. The proof is relegated to the appendix. The term "operational form" is used instead of the more customary word "transform." Although the operational forms of the derivatives of functions with arbitrary initial conditions are derived, very little use is made of them except in the case where the function and all its derivatives vanish when t is zero. The shift operator is neatly applied to obtain Fourier expansions in special cases. A treatment of ordinary differential equations with constant coefficients, Heaviside's unit function, and impulses closes the second part.

The technical problems treated in the third part are too numerous to mention. A feature here is the thorough treatment of several problems with the attainment of numerical results. This is after all the final aim of the technologist. The problems involving partial differential equations are not as clearly stated as those connected with ordinary derivatives.

A final criticism of omission must be made. The central theorem known as the "faltung" or convolution theorem is not even treated as a step-child. This theorem is known to the engineer as Borel's theorem. Under this name it receives only passing attention. The publication of this book is a step in the right direction and the author is to be congratulated for having written a book which will be very valuable to the mathematical technologist.

SAMUEL SASLAW

Théorie Mathématique du Bridge à la Portée de Tous. By Émile Borel and André Chéron. Paris, Gauthier-Villars, 1940. 17 + 392 pp. Fr. 175.

Here is a detailed study of shuffling, card distributions, the finesse, and other points of the game of bridge by an eminent mathematician and an authority on bridge. A survey of the context naturally enough begins with their initial discussion of the shuffling of the deck.

Shuffling of type A is that generally used in which the deck is separated into two approximately equal parts and then dove-tailed together in small alternating packets. Let a sequence be defined as two adjacent cards in an initial order. If the average number of sequences in the deck broken by one operation of A is S and N is the number of such operations, then NS greater than 150 will result in a well shuffled deck. Another test of the adequacy of shuffling is the fact that the mean number of unicolor sequences (two adjacent cards of the same color) is 12. Shuffling of type B: A packet of a_1 cards is

taken off the top of the deck. Then a_2 more are placed on top of the a_1, a_3 more at the bottom, etc. In terms of breaking up sequences, both methods can give satisfactory results, but B more slowly than A.

The authors' conclusion in this first chapter on card shuffling is that elementary shuffling of a deck of cards in the time allotted while another person deals the second deck, produces effectively a random placement of the individual card. It is to be noted that far short of this produces random composition of the four hands, i.e., a random residual of the ordinal position of a card modulo 4. In fact, it is observed by the authors that simply gathering up the tricks at the end of play in a jumbling fashion vitiates the most grave consequences of the absence of shuffling.

They present an interesting method of calculating the probability of a certain side (côté) having a given 26 cards (p. 56) and this is extended (p. 62 ff.) to calculating the probability of the four hands being of specified compositions. The method uses a small table of values constructed for the purpose and the probability of a single combination such as the 4333, 3433, 3343, 3334 distribution at specified positions. This is called the "method of coefficients." Chapters one and two deal with the a priori probabilities and mathematical expectations of the hands and sides. Seventeen tables are included in this part. A sample table is 24 which gives for specified numbers of cards of a suit in a hand the probability that a blank suit, blank suit or singleton, blank suit or singleton or doubleton are present in at least one of the adversaries' hands. The authors point out that some of the tables probably have results counter to the usual notions of what is believed by card players. A case in point is illustrated in Table 25 displaying the percentage frequency of non-occurrence of "accidents"—that is, blank suit or singleton. The accident is the rule for a side (two hands) rather than the exception.

There is a study of the defensive value of hands against bids of the opposition. From a study of the tables on strengths of long-suited hands for offensive and defensive games the conclusion is obtained that long-suited hands are very strong on the offense and very weak on the defense. On the other hand, the possessor of high cards without a long suit is better on the defensive. Chapter 3 also includes many illustrative situations showing how many tricks on an average the first lead (entame) from a suit such as $KQ7xxx$ will lose the leading side.

It is emphasized that the abnormal distribution of one suit, for example, a hand with 9 spades, has no effect in giving an expectation of a consequent abnormal distribution of one of the other suits (p.

159). On the other hand, the "law of attraction" between long and short suits and its converse the "law of repulsion" between longs (and shorts) are real aids in, say, locating a given card in two hidden hands. Thus if west has indicated a long suit in diamonds and five spades including a queen are in the two hidden hands, east will have on the average a longer suit in spades than west and consequently the queen of spades more often than west. Exact probabilities of such situations are given in Tables 72 to 101.

Many of the tables of the book are followed with brief summarizing statements which reduce their implications to a working "rule of thumb." For instance the summary of one table (111 bis) reads: "The bidder ought to run a practically unlimited risk (in number of tricks set—with probability of success equal to $\frac{1}{2}$) in order to try to make a bid of game or slam—with the two following exceptions. If it is a question of a game bid redoubled, or of a game bid doubled when the bidder alone is vulnerable, the failure ought never to exceed three tricks." There is considerable discussion of the conditions under which to choose the safe or risky line of play based on the mathematical expectation of the two methods. The question of whether or not to double and redouble is similarly taken up.

The notes comprising the last 128 pages discuss with more refinement some of the questions raised in the book proper. Note 1 considers the theoretical displacements of points in a line segment by random interchange of randomly chosen segmentations of the line. This is then compared to the similar but discontinuous case of shuffling cards. The observation is made that the displacements (i.e., shuffling) are in actuality better than that of the idealized cases studied. For one thing, instead of an independent sequence of "elementary operations"—choosing two random points and interchanging the three segments—in shuffling a deck of cards by the usual method of half the deck in each hand, the two points are replaced by many points. There is a consequent diminishing of the number of sub-segments (cards) not displaced, for the same total number of division points used.

Note 2 goes into a detailed study of the method used in enumerating the permutations of a given set of four partitions of 13. Thus the four hands might be $AAAB$ where A is the hand (3334) and B is (4441). From these values are obtained the total probability of a given designated deal. Perhaps the most interesting idea to the reader familiar with probability concepts is that of "virtual" probability discussed in Note 3. For the case of dependent probabilities the product rule holds for virtual probabilities of the separate events, as it holds

for real probabilities of independent events. The specific situation is: Consider N objects of K types so that $N=A_1+A_2+\cdots+A_K$. Choose by chance n of them. The probability that the composition of n is $a_1, a_2, \cdots, a_K$ of the K types is given by

$$P_{a_1, a_2, \cdots, a_K} = p_1 p_2 \cdots p_K$$

where

$$p_i = C_{A_i}^{a_i} / (C_N^n)^{A_i/N}.$$

p_i is the "virtual" probability that a_i of the A_i objects will be selected. This idea is applied to the selection of bridge hands and is the means of easily calculating through tables (114, 115) the real probability of one or more hands being of specified composition.

Note 4 gives a short discussion on the theory of finesse and on the "squeeze" play, with the observation that actual play generally amounts to choices between the advantages and disadvantages of the finesse and the squeeze play for the hands under consideration. A lengthy discussion with illustrations on the variation of probabilities in the course of play is the province of Note 6. The new probabilities are of course obtained by the use of Bayes' theorem. From the bidding and one or two rounds of play, certain restrictions can be deduced on the distribution of residue of a given suit. For purposes of finesse, the probabilities pertaining to positions of certain high cards must then be revalued as just indicated.

One of the most interesting tables is 129 which lists in order of decreasing frequency the distribution possibilities of the two hidden hands after the dummy has been displayed. The order is arrived at by a method explained in Note 7 as the "complement to the method of coefficients" mentioned previously. Some common errors of reasoning due mostly to misapplications or non-applications of Bayes' formula are elucidated in Note 8. The use of "psychological" probabilities for complete use of the formula is also given here. The final note of the book deals at length with a problem of finesse which again involves psychological probabilities, i.e., how the individual player reacts to given situations (e.g., can he be depended upon to cover the queen with the king of the suit, or to play the lowest card when following suit with a weak card), as well as information obtained from the cards already played.

It may be said that the book is written with the logically minded rather than the mathematically equipped person continually in view. The 134 tables involve a prodigious amount of computation. The fact that the calculating machine with automatic retransmission of

the product for a second multiplication was used instead of factorial and logarithm approximations, and that all probabilities were computed in at least two different ways, leads the authors to make highly confident statements as to the accuracy of the figures.

The reviewer's reaction to a first reading of this work was mainly a sense of increased appreciation of the fundamentals and potentialities of the game—much as one gets from a study of a good chess book. Further study of some of the vital points would undoubtedly save many a trick and generally improve the game of most of us.

JOSEPH A. GREENWOOD

Introduzione al Pensiero Matematico. By Friedrich Waismann. Translated by Ludovico Geymonat. (Biblioteca di Cultura Scientifica, vol. 111.) Torino, Einaudi, 1939. 325 pp. Lire 20.

This book, originally published in German (Vienna, 1936) and now translated into Italian, is a popular exposition of the fundamental concepts and views underlying modern mathematics. Special attention is paid to those sides of present day mathematics which may seem to involve "philosophical" problems, and a manifold of philosophical suggestions are advanced, mostly credited to an unpublished manuscript by Wittgenstein. The expository part of the book combines, to a rare degree, accuracy and comprehensibility.

After having traced in outline the historical growth of our present numbers system, the author envisages the question, often raised by philosophers: "How can the postulation of 'new' numbers be justified? Do there 'exist,' e.g., any irrational and complex numbers?" It is shown that these questions are equivalent to the questions whether the enlarged mathematical calculus is consistent, and whether it can be given an interpretation. The answer by a reference to geometrical facts is rejected as unsatisfactory, since proof of the consistency of geometry will depend on the assumption that our number system is consistent. The problem of the existence of the non-natural numbers —as far as it can be separated from the consistency problem—is now solved by giving the well known construction of them on the basis of the natural numbers. Following Skolem, the author then shows how elementary arithmetic can be strictly developed on the minimum basis of three undefined notions (natural number, successor, identity) and one inductive definition (of addition) by using the method of complete induction. The present situation of the "Grundlagenforschung" is reviewed, mention being made of theorems by Gödel and Skolem.

The "philosophical" ideas, expressed in the book, are unfortunately

presented in such a condensed form that it is difficult to judge their significance. I will here make a few critical comments, which should in no way obscure the high value of the book as an elementary exposition of the foundations of modern mathematics.

Throughout, the author stresses the important distinction between mathematics as a symbolic calculus and the various possible interpretations of this calculus. A strict observation of this distinction, however, should have led to more careful statements at a few points. When talking "philosophically" about what, for example, the natural numbers *are*, our statements must always be referred to some particular interpretation of arithmetic. But, in Chapters IV and V, it is said unconditionally that natural numbers, integers, and rational numbers *are* of three different logical types. This contention is, of course, true if we presuppose—as is done in the present book—the modern construction of integers and rational numbers. No reasons, however, have been given why there should not exist interpretations of another kind, and perhaps one which at once would give the natural numbers their usual meaning and make natural numbers, integers, and rational numbers all belong to the same logical type.

In Chapter VIII, the method of mathematical induction is discussed and a philosophical interpretation, due to Wittgenstein, is presented. This interpretation may be summarized thus. The sentence "Every (natural number) a has (the property) P," means, "1 has P, and if b has P, then $b+1$ has P." Following Skolem the author has not made use of the operators "for every x" and "for some x" in his development of elementary arithmetic. What he wants to interpret must, therefore, be the meaning of a statement, "a has P," where generality is expressed by the free variable a. The circularity of his explanation is then obvious. The positivistic principle of verification is invoked as a substantiation. This principle, as applied by the author to mathematics, could, I think, be formulated somewhat like this: "The meaning of a theorem p is identical with (the meaning of) the conjunction of those theorems which constitute the proof of p." This principle is open to a number of more or less obvious objections. If the words within parentheses in the above formulation are omitted, then it boils down to a purely syntactical definition which does not have anything to do with "meaning" in an interpretative sense. Whether the author would let them stand or not, cannot, however, be gathered from his presentation. But further: The principle leads to the consequence that only provable sentences have meaning, a consequence which seems absurd if "meaning" is taken in the interpretative sense. And what is "*the* proof" of a mathematical theorem? From the prin-

ciple of verification, by the way, several other conclusions are drawn. One of them is the Brouwerian contention that the negation of a universal sentence is not equivalent to an existential sentence. This conclusion is clearly possible only if we are already arguing on the basis of an axiomatization of mathematics in accordance with Brouwer's view!

In Chapter IX: B the Frege-Russell interpretation of natural numbers is criticized. The discussion is centered around their well known definition of the phrase "the number of elements in the two sets A and B is the same" as meaning "there is a one-one relation of which A is the domain and B the converse domain" ("A"). The chief objections made against this definition are: (1) According to the definition, the number of elements in two sets A and B must either be the same or not be the same, whereas in reality there are sets A and B such that neither the sentence "A and B have the same number of elements" ("B") nor its verbal negation ("C") is true, if we take this sentence with its accepted meaning. A contention like this, of course, cannot be conclusively proved or refuted. But the alleged instance of two such sets, mentioned by the author (p. 156), does not seem convincing. The argument is essentially dependent on the obscure principle of verification, here applied to empirical sentences. However, even if the contention were true, it is still not clear in what way it would be an objection to the Frege-Russell analysis. The contention seems to allow two different interpretations: (i) By C, the verbal negation of B, we ordinarily understand the logical negation of B; but the law of the excluded middle is not universally valid. (ii) When B and C, as ordinarily used, are meaningful, then C is the logical negation of B; but in the case at hand—by force of the principle of verification —they are both deprived of meaning, and therefore the law of the excluded middle is not relevant. If (i) covers the author's intentions —why could not A, too, be a sentence for which the law of the excluded middle does not hold? If (ii) is meant—why could not the principle of verification sometimes make A and non-A meaningless, too? (2) Two sets, A and B might have the same number of elements —the phrase being taken in its usual meaning—in spite of the fact that no one-one relation held between them; the occurrence of such a relation is an accidental empirical fact. Here the term "relation" has obviously been taken in a restricted sense as synonymous with "physical relation," a sense not intended by the Frege-Russell theory. (The question has been discussed by Ramsey in his *Foundations of Mathematics*.)

In a concluding chapter (to be compared with Chapter IX: C), the

philosophical standpoint of the book is restated. In opposition to the view that the rules governing the use of a sign follow from the meaning of the sign, it is asserted that the sign obtains its meaning just through the rules. The rules the author has in mind seem to be of the following kinds: (1) the syntactical rules of the mathematical calculus within which the sign occurs; (2) the rules for translating the mathematical formulae into our everyday language (e.g., "two plus two make four" is a translation of "$2+2=4$"); (3) the rules for integrating the mathematical calculus into everyday language (e.g., from "$2+2=4$" we may deduce "two apples plus two apples make four apples"). Given these rules, the meaning of "1," "2," and so on is determined. To that extent, the author's contention that "the rules give meaning to the signs" is obviously true. The words "one," "two," and so on are, in their everyday usage, endowed with a certain meaning, and by the rules, belonging to category (2), this meaning is transferred to "1," "2," and so on. But, on the other hand, if we let "1," "2," $\cdots$ mean what is ordinarily meant by "one," "two," $\cdots$, then we cannot arbitrarily change the rules of categories (1) and (3) and still obtain true mathematical formulae and valid deductions. So far, the view which the author rejects seems to be a commonplace. Possibly, the author would add to the above list of rules a fourth category: rules for deciding when empirical numerical sentences are true. Through these rules, he would say, words like "one," "two" get their meaning, and thus, ultimately, rules of usage constitute the meaning of mathematical terms. But—can we give any rule for the use of the term "two" that is not of the following type: "If and only if the kind A has the property P then the sentence 'There are two A's' is true"? If, as it seems to be, we cannot, then we have reduced the meaning of "two" to the meaning of "P" which latter meaning we have not explained in terms of any rules of usage.

It might be worthwhile mentioning that the theory of natural numbers, given in Chapter VIII, suffers from a somewhat subtle logistic fault. Letting his variables range over a domain of objects, called "natural numbers," the author takes the monadic successor function $a+1$ as his undefined concept besides the relation of identity. On this basis he gives the following "inductive" definition of the dyadic additive function: $a+(b+1)=(a+b)+1$. To be in the standard from, this definition should be completed by another equation: $a+1=\cdots$. The omission here suggests a confusion of the successor function $a+1$ (argument a) with the additive function $a+1$ (arguments a and 1).

ANDERS WEDBERG

Reelle Funktionen, Part I. By C. Carathéodory. Leipzig, Teubner, 1939. 6+182 pages. RM 11.20.

This volume is Part I of a projected three volume treatise on real functions to be distributed under the following titles: Part I, Numbers, Point Sets, Functions; Part II, Theory of Measure and The Integral; Part III, as yet without a title, but presumed to treat of derivatives and applications.

These volumes will replace the author's one volume, *Vorlesungen Über Reelle Funktionen*, first completed in 1917 and later reissued in 1927 with slight changes. It will be recalled that this earlier volume is written in the elegant axiomatic style for which the author is noted. Among its outstanding features are an abstract treatment of measure and a very full treatment of Lebesque integration.

This volume preserves much of the material in the first 230 pages of the older treatise. A considerable rearrangement of material has been made which adds to the attractiveness of the book. But examples are given only when required to show the necessity for the conditions governing certain theorems. The reviewers believe that the addition of illustrative examples would have been appreciated by many readers.

Some of the essential changes and additions will be taken up by chapters.

Chapter I—Real Numbers. Denumerability and non-denumerability treated in Chapter II of the older treatise here follows an axiomatic introduction to real numbers.

Chapter II—The Limit Notion. A discussion of Cantor's diagonal process is added to the older chapter on limits.

Chapter III—Point Sets in Euclidian Space. The treatment of open and closed sets also involves an iteration process of forming closures and interiors. An application is made to boundary theory.

Chapter IV—Normal Covering Sequences and the Theory of Connectedness. A new feature is the treatment of connectedness of arbitrary sets.

Chapter V—Functions. No essential change over the older volume save for omission of a discussion of functions of bounded variation (deferred to Part II) and the inclusion of a section on continuous monotonic functions in which is introduced a transformation function

$$x = \lambda(y) = \frac{1+y+|y|}{2(1+|y|)}$$

which sends the y-interval $(-\infty, \infty)$ into the x-interval $(0, 1)$. This preserves local limit properties and permits the extension of some theorems on positive bounded functions to more general functions.

Chapter VI—The Distance Function and Applications. The chapter closes with Radó's proof of a theorem on the extension of closed domains of definition of functions. A different proof was given much later in the older volume.

Chapter VII—Sequences of Functions. This chapter contains perhaps the most interesting addition to the corresponding material of the older volume; namely, continuous convergence and normal function families. The theory of continuous convergence is developed in an elegant manner. However, the author might have pointed out the close connection existing between this theory and ordinary continuity of functions. By the simple device of considering the subscript k of the sequence $f_k(P)$ as another coordinate in our number space, continuous convergence of the sequence becomes identical with the existence of a limit at the point (P_0, ∞) of the corresponding function $F(P, k)$ in the extended space. Thus, for example, Theorem V follows immediately when it is noted that the two conditions given there are equivalent to the requirement that the saltus of $F(P, k)$ vanish at (P_0, ∞), while Theorem VII follows directly from the theorem: A function continuous on a bounded and closed set is uniformly continuous on that set.

The final section of the chapter, on normal families of functions, while brief, contains recent results on function sequences of great interest to the advanced student of analysis.

Some readers will find it regrettable that the author retains in this modern work his limitation of the discussion to euclidean spaces, thus leaving out of account the far-reaching extension of real function theory to more general spaces made during the last thirty years; the more so, as much of the theory can be given in this general setting without in any way complicating the reasoning or lengthening the proofs. Bohnenblust's 1938 *Princeton Notes on Real Variables* indicates the possibilities of such a treatment.

Since this volume contains the more elementary material many readers will naturally be more interested in Parts II and III. The known high standards of the author are enough to insure the completed work a prominent place in mathematical literature.

J. W. HURST
H. M. SCHWARTZ

A History of Geometrical Methods. By J. L. Coolidge. Oxford, Clarendon Press, 1940. 18+452 pp. $10.00.

Notwithstanding the rich literature on geometry, there has been no comprehensive synopsis of the subject for a long time. The author has set himself the task of presenting in outline the development of geometry from the beginnings to the present, emphasizing in particular the various methods employed, taking for a general model Chasles's *Aperçue Historique*, which first appeared in 1837. The field is so large that not all material could be considered; selections must be made, and no fixed criterion can be set down for eligibility.

The book is divided into three parts: synthetic, algebraic, and differential. It is explicitly stated that topology is excluded from consideration. The first part covers 116 pages. It includes a sketchy summary of practically everything preceding Descartes, all of synthetic projective geometry, non-euclidean geometry in the narrower classic sense, and a partial discussion of descriptive geometry; axonometry is not mentioned, and the treatment of perspective is very brief. Pohlke's theorem is not mentioned. The substance of this part is largely considered as a completed edifice.

The second part, algebraic geometry, covers 200 pages. It is the field of most of the author's own creative activity; many topics are referred to his extensive writings for fuller discussion. After pointing out various usages of coordinates in earlier writings, the classic period of Fermat and Descartes is given in some detail, followed by a discussion of abridged notation, change of space element and extension to higher dimensions. This in turn is followed by a short description of the Clebsch-Aronhold symbolic notation, minimal coordinates, of elliptic coordinates, pentaspherical coordinates, and other systems.

Unusual care is devoted to the study of enumerative geometry, particularly to the Schroeter calculus. The opinion is suggested that this subject has provided more errors in the literature of the subject of algebraic geometry than any other. After a stormy and acrimonious history, the principles of the discipline are now fully established, but in order to apply them properly, the operator must know how to count. The closely allied theory of correspondence between points of associated algebraic loci is discussed from the same point of view.

The chapter on birational geometry furnishes a good introduction to the geometry on algebraic curves and surfaces; rather more emphasis is put on the transcendental treatment than the history of the literature of the subject warrants. The treatment of the uses of higher spaces is well written, but too brief to be of maximum usefulness, except possibly the paragraph on quaternions. The chapter on trans-

formations begins with linear ones (both collineations and correlations), then quadratic ones and the general Cremona transformation. In the plane, this is practically complete. Like synthetic geometry, this subject involves a number of indispensable tools, the use of which must be thoroughly mastered for further progress, without the hope of contributing much from this section that is new. On page 288 is the remark that Mlodziejowski computed the tables of plane Cremona transformations to order 21. Without debating the real merits of this work, the reviewer inclines to the opinion that the Naples dissertation (1909) of Marazzo should also be mentioned, as it provides the tables of orders up to 23.

After a careful and correct discussion of Hudson's fundamental theorem that in space of more than two dimensions a theory of composition analogous to Noether's theorem for the plane does not exist, the author dismissed the theory in more dimensions with the implication that there isn't much there. In the sense of results, this is correct. The author ascribes this state of affairs to the greater difficulty and to the lack of general methods. What greater incentive is needed? During the last few years the question of the existence of irrational involutions of order two in three way space has been discussed in over a hundred papers, and that of a possible new approach to the question of composition has a literature that is nearly as large. It is pointed out that the group concept will probably play an important part in further advances along this line.

The third and last part of the book, differential geometry, covers 100 pages. After a brief glance at the early writers, including Euler and Monge, a detailed exposition of the methods of Cauchy follows, then the use of intrinsic methods and of movable axes. A full analysis of the work of Gauss is given; it plays an important part in much of the later metric differential geometry. Geodesic lines on a general surface are treated, followed by a summary of the essential property of minimum surfaces. A brief discussion of differential properties of line congruences is then given.

The chapter on projective differential geometry compares the methods of Wilczynski, Fubini, Cartan, and their followers. The name Halphen is not mentioned in this connection, but he was more truly a founder than the others. The range of problems to which these methods are applicable is sharply limited, but within these limitations, important results are obtained with ease and elegance.

The last chapter, on absolute differential geometry, is devoted largely to an explanation of the notation employed. This is not entirely consistent, and the pace is so rapid that its usefulness remains

in doubt. Riemannian geometry, parallelism and the geometry of paths are all touched upon.

The book is provided with a subject index and an extensive bibliographical one. The latter includes (most of) the names cited, including the title and a reference to its source, and in many cases the date of birth and of death of those cited. This is a valuable list; its compilation is a difficult and often a thankless task. A real source of confusion arises in some cases in which the information is not complete, especially when the person cited was born after the middle of the nineteenth century. A special symbol to indicate that the entries were not complete in such cases would have avoided the ambiguity. Numerous slips or actual errors were noticed. Six proper names are misspelled; for homaloid the spelling homoloid is everywhere used. In the formula at bottom of page 221 for i read $i-1$. A real source of confusion arises on page 220 line 10 up. The genus defined is not the geometric genus, but the number p' in the author's notation. The discussion on the following page also refers to p'. This is what Noether called the Curvengeschlecht.

On page 208 is a footnote concerning the origin of the transformation connecting adjoints of plane curves, and hyperspace. Compare Cayley: *On the transforms of curves*, Proceedings of the London Mathematical Society, vol. 1 (1865). The formula $I+p-\sigma=12p_a+9$, in which I is the invariant of Zuethen-Segre and σ the number of exceptional curves in the system, is discussed in the footnote page 227. Compare Noether, Mathematische Annalen, vol. 8 (1875), pp. 495–528.

Professor Coolidge has put a great deal of careful thought in the preparation of this book. A number of concepts have been correctly explained, which have been sources of confusion in the minds of many, especially those interested primarily in other branches of mathematics. The book will serve a real purpose.

VIRGIL SNYDER

Festschrift Rudolph Fueter zur Vollendung seines sechzigsten Altersjahres, 30 VI. 1940. Zürich, Naturforschende Gesellschaft, 1940. 231 pp.

This volume comprises articles by Oystein Ore, Henri Lebesgue, M. Plancherel, N. Tschebotarëw, Paul Montel, W. Scherrer, L. J. Mordell, Francesco Severi, T. Carleman, E. Hecke, H. S. Vandiver, R. Wavre, H. Brandt, C. Carathéodory, Heinrich Jecklin, Eugenio G. Togliatti, Alfred Kienast, Ernst Trost, Ludwig Bieberbach, J. J.

Burckhardt, Paul Finsler, Heinz Hopf, H. Behnke and K. Stein, Elie-Cartan, Andreas Speiser, Max Gut, F. Gonseth.

W. W. FLEXNER

Les Probablitités Associées à un Système d'Événements Compatibles et Dépendants; I. *Événements en Nombre Fini Fixe.* By Maurice Fréchet. (Actualités Scientifiques et Industrielles, no. 859.) Paris, Hermann, 1940. 8+80 pp.

This is part one of a series of three, the others being: II. *Cas Particuliers et Applications*, and III. *Evénements en Nombre Très Grand ou Infini*. In this series Professor Fréchet has gathered together the hitherto scattered literature on a problem of rather general interest. The problem may be stated thus. We consider m quite general events $A_1, \cdots, A_m$ and an event H which is a function of these; that is, the occurrence or non-occurrence of H depends solely on which of the A's occur. The probability that $A_{i_1}, \cdots, A_{i_r}$ occur simultaneously is denoted by $p_{i_1 \cdots i_r}$. We wish to find the probability of H, granted that we know the values of the p's.

In Chapter I the author states and proves two interesting and powerful theorems due to Broderick. The first of these theorems asserts that the probability of H is a linear function of the p's, with coefficients depending not on the particular nature of the A's, but only on the function H. In the second theorem it is shown that if H is considered to be a function of two sets of events, then the probability of H can be obtained by a symbolic multiplication. The utility of these theorems in obtaining elegant solutions of certain classical problems will no doubt be demonstrated in the second volume of the series.

Chapter I also contains certain related formulae on moments, generating functions, and "conditional" probabilities. In Chapter II the author obtains a number of inequalities due to various writers, some in generalized form. In the remainder of the chapter questions of the following type are answered: what are the necessary and sufficient conditions that a set of 2^m numbers be the probabilities $p_{i_1 \cdots i_r}$ defined above, for some set of events $A_1, \cdots, A_m$?

The mathematics used throughout is on a quite elementary level so that the book should prove of interest to a wide circle of readers. A defect, in the reviewer's opinion, is the seemingly haphazard manner in which the various topics are arranged.

I. KAPLANSKY

NOTES

The editors of the Bulletin wish to make grateful public acknowledgment of the services rendered by the following persons who have refereed papers: R. P. Agnew, Leonidas Alaoglu, Reinhold Baer, R. W. Barnard, I. A. Barnett, Walter Bartky, P. O. Bell, A. A. Bennett, Garrett Birkhoff, G. D. Birkhoff, Henry Blumberg, R. P. Boas, Salomon Bochner, A. T. Brauer, Richard Brauer, H. W. Brinkmann, A. B. Brown, E. W. Chittenden, Alonzo Church, R. V. Churchill, J. A. Clarkson, A. H. Clifford, A. B. Coble, George Comenetz, A. H. Copeland, C. M. Cramlet, M. M. Day, J. J. DeCicco, Nelson Dunford, Ben Dushnik, L. R. Ford, Orrin Frink, Guido Fubini, H. L. Garabedian, J. J. Gergen, D. W. Hall, W. L. Hart, M. R. Hestenes, Einar Hille, Witold Hurewicz, Nathan Jacobson, R. D. James, R. L. Jeffery, B. W. Jones, Mark Kac, E. R. van Kampen, Edward Kasner, S. C. Kleene, Fulton Koehler, E. P. Lane, C. G. Latimer, D. H. Lehmer, D. C. Lewis, E. R. Lorch, N. H. McCoy, J. C. C. McKinsey, E. J. McShane, Saunders MacLane, H. W. March, Morris Marden, Karl Menger, G. M. Merriman, Deane Montgomery, C. B. Morrey, A. F. Moursund, Oystein Ore, W. F. Osgood, Gordon Pall, G. H. Peebles, F. W. Perkins, B. J. Pettis, Hillel Poritsky, H. A. Rademacher, Tibor Radó, W. C. Randels, A. Rappoport, W. T. Reid, J. F. Ritt, J. H. Roberts, M. S. Robertson, R. M. Robinson, A. E. Ross, J. B. Rosser, O. F. G. Schilling, I. J. Schoenberg, Wladimir Seidel, W. E. Sewell, I. M. Sheffer, J. A. Shohat, L. L. Smail, I. S. Sokolnikoff, N. E. Steenrod, Alvin Sugar, Otto Szász, Gabor Szegö, J. H. Taylor, H. P. Thielman, T. Y. Thomas, J. W. Tukey, S. M. Ulam, J. V. Uspensky, R. J. Walker, A. D. Wallace, J. L. Walsh, L. E. Ward, M. S. Webster, Hassler Whitney, Norbert Wiener, L. R. Wilcox, John Williamson, Aurel Wintner, Leo Zippin, and Max Zorn.

The editors of the Transactions wish to acknowledge the services of the following persons, not members of the Editorial Board, who have been consulted regarding papers offered for publication in volumes 47 and 48: W. L. Ayres, Reinhold Baer, Walter Bartky, Garrett Birkhoff, Salomon Bochner, H. R. Brahana, A. B. Brown, H. E. Buchanan, A. B. Coble, H. S. M. Coxeter, R. P. Dilworth, J. L. Doob, Nelson Dunford, W. K. Feller, G. A. Hedlund, J. R. Kline, C. G. Latimer, N. H. McCoy, J. C. C. McKinsey, R. E. von Mises, S. B. Myers, B. J. Pettis, Tibor Radó, W. T. Reid, I. J. Schoenberg, Wladimir Seidel, D. J. Struik, Otto Szász, J. D. Tamarkin, Alfred

Tarski, J. M. Thomas, A. W. Tucker, J. W. Tukey, G. E. Wahlen, H. S. Wall and Aurel Wintner.

The National Research Council has just issued a report on the *Bibliography on Orthogonal Polynomials* which was prepared by a committee consisting of Professors J. A. Shohat, Einar Hille and J. L. Walsh. The report is bound in cloth, consists of 204 pages and may be purchased from the National Research Council at $3.00 per volume.

The addresses presented at the symposium on Applications of Mathematics in the Earth-Sciences, under the joint auspices of Sections A and E of the American Association for the Advancement of Science, the American Mathematical Society, the Mathematical Association of America and the Geological Society of America, at Columbus, Ohio, December 29, 1939, have been published as Part A of Volume 4 of the Transactions of 1940 of the American Geophysical Union. The addresses tend to further a desirable rapprochement between geophysicists and mathematicians and begin the important task of bringing to each field the knowledge or some suggestions of the contributions which the other field may make. This volume will include also, as Part B, the symposium at Columbus on Hydrologic Problems in the Ohio and Michigan Basins. The volume will make about 100 pages. The regular postpaid price to non-members of the Union is $1.00 but a special price of $0.75 per copy is being made to members of the American Mathematical Society. Orders, indicating membership in the American Mathematical Society, should be sent to 5241 Broad Branch Road, N.W., Washington, D.C.

Professor M. H. Stone of Harvard University has published a planographed edition of a set of notes on *The Theory of Real Functions*: I, *The Real Number System*. II, *The Theory of Classes*. Interested persons should address Professor Stone.

Professor Saunders MacLane of Harvard University has published a planographed edition of a series of lectures on *Algebraic Functions*, which may be obtained from him by interested persons.

Applications for the Benjamin Peirce Instructorships in Mathematics at Harvard University for the year 1941–1942 should be sent to the Chairman of the Department of Mathematics. Candidates should have received the doctorate or have done equivalent work.

The School of Mathematics of the Institute for Advanced Study each year allocates a small number of stipends to gifted young mathe-

maticians and mathematical physicists to enable them to study and to do research work at Princeton. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks for application may be obtained from the School of Mathematics of the Institute, Fuld Hall, Princeton, N. J., and are returnable by February 1, 1941.

Yale University offers each year a number of Sterling research fellowships, some of them open to mathematicians. For the year 1941–1942 the normal stipend will be $1800. Applicants must have received the degree of Doctor of Philosophy or its equivalent from approved universities in the United States or foreign countries. In general, preference is given to applicants of not more than thirty-five years of age, but older persons are eligible. Applications should be received before March 1; blanks may be obtained from the Graduate School office, Yale University.

Dr. K. E. Bullen, lecturer in mathematics at University College, Auckland, New Zealand, has accepted an appointment at the University of Melbourne.

Assistant Professor G. Hoheisel of the University of Greifswald has been appointed to a professorship at the University of Köln.

Dr. F. Lösch of the University of Rostock has been promoted to a professorship.

Professor R. P. Paranjpye of Fergusson College, Poona, India, has retired.

Dr. W. B. Caton has been made head of the department of mathematics at Athens College, Athens, Alabama.

Dr. Claude Chevalley, formerly of the University of Rennes, has been appointed to an assistant professorship at Princeton University.

Dr. P. L. Dressel of Michigan State College has been promoted to an assistant professorship.

Associate Professor E. P. R. Duval of the University of Oklahoma has been promoted to a professorship.

Mr. W. O. Gordon of Pennsylvania State College has been promoted to an assistant professorship.

Assistant Professor Nathan Jacobson of the University of North

Carolina has been appointed visiting associate professor of mathematics at Johns Hopkins University this year.

Dr. J. R. Jenness of the College of the Ozarks has been made head of the department of physics and mathematics at Parsons College, Fairfield, Iowa.

Dr. S. A. Jennings of the University of British Columbia has been appointed lecturer.

Assistant Professor H. A. Jordan of Georgetown University has been promoted to an associate professorship.

Professor L. C. Knight of the College of Wooster has retired after thirty-two years of service.

Mr. T. H. Lee of the University of Wisconsin has been appointed to an associate professorship at the University of South Carolina.

Assistant Professor A. N. Lowan of Yeshiva College has been promoted to an associate professorship.

Associate Professor Morris Marden of the University of Wisconsin at Milwaukee has been granted leave of absence for this academic year. He is lecturing at the University of Wisconsin in Madison.

Assistant Professor W. T. Martin of the Massachusetts Institute of Technology has been granted leave of absence to spend a year in study and research at Princeton University and the Institute for Advanced Study.

Dr. R. J. Michel of the University of Missouri has been appointed head of the department of mathematics at Southeast Missouri State Teachers College.

Professor J. C. Morehead of the Carnegie Institute of Technology has retired with the title emeritus.

Dr. L. F. Ollmann of Texas Technological College has been appointed to an assistant professorship at the College of Wooster.

Professor George Polya who has been at the Technical School in Zurich is to be at Brown University for two years.

Assistant Professor W. C. Randels of the University of Oklahoma has been promoted to an associate professorship.

Associate Professor H. A. Ruger of Teachers College, Columbia University, has retired.

Mr. C. S. Sutton of Tufts College has been appointed to an assistant professorship at The Citadel.

Commander J. C. Van de Carr, U.S.N., retired, has been granted a leave of absence from his duties in the mathematics department at the University of Oklahoma to take charge of the University unit of the Naval R.O.T.C.

Dr. E. T. Welmers of Michigan State College has been promoted to an assistant professorship.

Professor R. L. Wilder of the University of Michigan has been granted leave of absence to spend a year at the University of Texas.

To the list of names of members of the Society working at the Institute for Advanced Study this year, which appears in the November 1940 issue of this Bulletin, should be added those of Assistant Professor G. M. Ewing of the University of Missouri and Dr. H. T. Muhly, National Research Fellow.

Dr. R. J. Duffin of the University of Illinois has been made an associate in mathematics.

The following appointments to instructorships are announced: Bowling Green State University: Dr. Morris Hendrickson; Cornell University: Dr. Michael Golomb; Georgia School of Technology: Dr. D. T. McClay; Long Island University: Mr. E. A. Knobelauch; Massachusetts Institute of Technology: Dr. G. B. Thomas; Michigan State College: Mr. A. C. Cohen, Dr. B. M. Stewart; Montana State College: Dr. H. M. Schwartz; Northwestern University: Dr. J. M. Dobbie, Mr. W. N. Huff; University of Oklahoma: Mr. B. S. Whitney; Pennsylvania State Forestry School, Mont Alto, Pennsylvania: Mr. Morris Bloom; University of the Philippines: Dr. R. A. Favila; Purdue University: Dr. R. A. Leibler; Stanford University: Dr. P. V. Reichelderfer; Agricultural and Mechanical College of Texas: Dr. R. E. Basye; Texas Technological College: Mr. F. D. Rigby; Tri-State College, Angola, Indiana: Dr. M. G. Moore; University of Virginia: Mr. R. C. Morrow; Williams College: Mr. W. D. Wray; Winthrop College, Rock Hill, South Carolina: Miss Katharine E. Hazard; University of Wisconsin: Dr. H. E. Goheen, Dr. Fred Kiokemeister, Mr. J. W. Odle; University of Wisconsin at Milwaukee: Mr. W. L. G. Mitchell; University of Wyoming: Dr. V. J. Varineau.

Dr. Vito Volterra, formerly professor of mathematics at the Uni-

versity of Rome, died October 11, 1940 at the age of eighty years. He had been a member of the Society since 1905.

Dr. J. R. Carson of the Bell Telephone Laboratories died on October 31, 1940. He had been a member of the Society since 1918.

Professor George Rutledge of the Massachusetts Institute of Technology died on September 21, 1940, at the age of fifty-eight years. He had been a member of the Society since 1910.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA

1. R. H. Bruck: *The structure of the rational representations of a wide class of linear groups.*

Let θ, $\phi=\theta^{-1}$ be fixed automorphisms of the underlying field (of characteristic 0 or p). Denote by A^* the transposed of the matrix obtained from A by subjecting its elements to the automorphism ϕ. Attention is restricted to groups $\mathfrak{G}$ of $n\times n$ matrices with the property that A in $\mathfrak{G}$ implies A^* in $\mathfrak{G}$; by choice of ϕ a large number of groups may be brought under review. $\mathfrak{G}$ being taken as a group of transformations of a contravariant vector x^i, the fth Kronecker product of $\mathfrak{G}$ may be regarded as a group of transformations of the linear vector space S of all forms $F(T)=\sum c_{i_1 i_2 \cdots i_f} T^{i_1 i_2 \cdots i_f}$, where $T^{i_1 i_2 \cdots i_f}$ is a contravariant tensor. The fundamental idea of the paper is to analyse the Kronecker product by means of a scalar product defined on S. The scalar product of F and $G=\sum d_{i_1 i_2 \cdots i_f} T^{i_1 i_2 \cdots i_f}$ is taken as $F \circ G \equiv \sum c^{\theta}_{i_1 i_2 \cdots i_f} d_{i_1 i_2 \cdots i_f}$, and has properties (1) linearity, (2) $(bF) \circ G = F \circ (b^{\theta} G) = b^{\theta}(F \circ G)$, (3) $F \circ G = F_{(A^*)} \circ G_{(A^{-1})}$. (Here $F_{(A)}$ is defined by $F(T) \equiv F_{(A)}(T')$ where $x=Ax'$.) The following special result indicates the nature of the paper: If $\mathfrak{G}$ (over the field of rationals) contains the transposed of each group matrix, the Kronecker product representations are completely reducible. (Received November 12, 1940.)

2. C. C. Camp: *A root cubing method of solving equations.*

Algebraic equations were solved by Dandelin and Graeffe by the well known root squaring process. Instead of squaring the roots the present paper gives a method for cubing the original roots. No Encke roots are needed and the sign of a real root is preserved on successive cubings. The general rule is derived by two independent methods and in the case of the cubic by three. The cases of complex and repeated roots are treated in the applications together with that of distinct real roots. (Received November 23, 1940.)

3. W. D. Duthie: *Segments in ordered sets.*

The notion of "segment" in a linearly ordered set is extended to (partially) ordered sets. Segments in ordered sets are self-dual, and can be used to characterize various special properties of ordered sets such as being a lattice, modular, distributive, or complemented lattice, giving a completely self-dual set of postulates for these lattices. Convex subsets of a lattice are defined in the natural way, and various relations among a lattice and its lattices of segments and convex subsets are discussed,

leading to results on imbedding of lattices in complete lattices. The lattices of segments of modular and distributive lattices are special cases of pseudo-modular and pseudo-distributive lattices, respectively; the latter property is of particular interest, since it is applicable to modular lattices, though the application is somewhat restricted by the fact that the only complete complemented pseudo-distributive modular lattices which are not distributive are one-dimensional projective geometries. However in such cases the existence of a dimension function permits replacing pseudo-distributivity by a more general notion of "m-distributivity" (m having the range of the dimension function). Application of the general theory of segments to Boolean algebras yields some known results by new and shorter methods. (Received November 25, 1940.)

4. C. J. Everett: *An extension theory for rings.*

A ring theoretic analogue of the O. Schreier "Erweiterungstheorie" for groups is developed about the central problem of determining all rings R containing a given ring N as ideal and such that R/N is isomorphic to a given ring F, employing factor systems and their equivalence. The Artin "splitting theorem" is proved: If R contains ideal N, R has a splitting ring over N. Constructive proof is given of the existence of rings whose chains of left and right annihilators for successive powers of the ring have first repetitions at arbitrary positions. "Closed rings" are shown to be direct summands of over-rings containing them as ideals. Equivalence classes of suitably restricted factor systems form an abelian group under addition. (Received November 23, 1940.)

5. Irving Kaplansky: *Maximal fields with valuations.*

A field K with a valuation is said to be maximal if no proper extension of K has the same value-group and same residue-class field as K. It was shown by Krull that any field with a valuation could be embedded in a maximal field, and he proposed the problem of determining whether this maximal extension is unique. This question is examined in the present paper, and the following result is obtained. The maximal extension is always unique if the residue-class field has characteristic zero; but if the latter has characteristic p, it must first be required that the value-group have no extensions of degree p, and second that the residue-class field satisfy a certain condition somewhat stronger than algebraic perfection. Under the same hypotheses it is shown that in the equal characteristic case every maximal field is a generalized type of power series field. The chief tool employed is a generalization of Ostrowski's notion of pseudo-convergence, in terms of which it is possible to give new criteria for maximality and completeness. (Received December 2, 1940.)

6. Rufus Oldenburger: *The minimal number problem.*

It has been proved that, for a field with at least $n+1$ elements, each binary form F of degree n with a symmetric tensor A of coefficients can be written as a linear combination of nth powers of linear forms. The least number of terms for which F can be so written is called the minimal number m of F. The elements of A can be arranged in ordinary matrix displays $B_0, \cdots, B_n$ when B_i is the display obtained from A by using i indices of A as column indices. The minimal number problem is the problem of determining the precise relation between the ranks of $\{B_i\}$ and m. The ranks of the matrices $\{B_i\}$ are determined by one of them, denoted by γ. In the present paper the minimal number problem is solved for algebraically closed fields by proving that $m=\gamma$ or $m=n-\gamma+2$. A modification of this result holds for arbitrary fields. The

proof is accomplished in part by the solution of the problem of construction of all representations of F. This latter problem is the same as the problem of constructing all forms apolar to F. Solutions for several major classical problems of algebra follow. (Received November 23, 1940.)

7. G. Y. Rainich: *Postulates for fields.* Preliminary report.

Instead of binary operations or functions of two variables, namely, the sum $x+y$ and the product xy, in terms of which a field is usually defined, corresponding functions of one variable or transformations, namely, addition $x'=y+a$ and multiplication $x'=xb$, are considered. The problem is to characterize the field by studying the formal properties of these or, rather, of the group $x'=xb+a$ generated by them. This group is postulationally defined in two ways; once as a two-fold transitive group of transformations and then again as an abstract non-abelian group. From the first point of view the additions and multiplications are introduced as transformations without fixed point and with a given fixed point respectively. This point of view is closely related to projective geometry. From the other point of view the subgroups of additions and multiplications are introduced as the commutator subgroup and its factor group respectively. (Received October 25, 1940.)

8. N. E. Rutt: *Rectangular arrays of combinations.*

Let C be the collection of different combinations, two at a time, of the objects in a set D of $2n$ different symbols. From the elements of C construct a rectangular arrangement, henceforth to be called an array, such that no element of D occurs twice in the same row of the arrangement, and no element of C occurs twice in the same arrangement. Two arrays are the same if and only if a reordering of their rows or a reordering of the members of C in their rows will make them formally identical. An investigation of the properties of arrays is undertaken in this paper. It is proved that, for each even natural number $2n$, there is an array whose rows contain n elements of C (the maximum possible), and whose rows are in number $2n-1$ (the maximum possible). Features of the internal structure of arrays are considered, among them the existence and inter-relations of the proper sub-arrays of arrays. The behavior of arrays under permutations of the elements of D receives some attention. An attempt is also made to determine the number of different arrays of certain specified types. (Received November 18, 1940.)

9. M. F. Smiley: *Measurability and distributivity in the theory of lattices.*

G. Birkhoff's criterion that a metric lattice be distributive (American Mathematical Society Colloquium Publications, vol. 25, p. 81) is quantified so as to provide a second generalization (this Bulletin, vol. 46 (1940), pp. 239–241) of the measurability of Carathéodory (*Vorlesungen über Reelle Funktionen*, 2d edition, p. 246). Closure properties of the set of "measurable" elements are derived as before. The influence of distributivity on measurability and on the measurability of complements of measurable elements is discussed. (Received November 20, 1940.)

10. T. L. Wade: *Tensor algebra and Young's symmetry operators.*

A. Young's symmetry operators (Proceedings of the London Mathematical Society, vol. 33 (1900), pp. 97–146) have recently been used by H. Weyl (*The Classical Groups*, Princeton, 1939, pp. 119–131) in the decomposition of tensor space into

irreducible subspaces. This paper is concerned with a study of the law of combination satisfied by the indices of the tensor associated with each diagram or partition of the indices of the arbitrary tensor. It is noted that corresponding to each partition there can be associated a specialization of C. M. Cramlet's general invariant tensor (Tôhoku Mathematical Journal, vol. 28 (1927), pp. 242–250), and that an arbitrary tensor can be decomposed by multiplying an appropriate numerical tensor identity by it, and then contracting. (Received November 25, 1940.)

11. R. W. Wagner: *The differentials of analytic matrix functions.*

The notions of Hausdorff and Fréchet concerning differentials are applied to analytic matrix functions as defined by the author in a previous paper. The differential of an analytic matrix function is shown to have a simple form when it is written in terms of the characteristic roots and partial idempotent and partial nilpotent elements of the argument. The function enters the differential through divided difference quotients. This simple form of the differential is used to deduce local properties of the mapping defined by the function. (Received November 23, 1940.)

12. Morgan Ward: *The fundamental theorem of arithmetic.*

A set of necessary and sufficient conditions for the fundamental theorem of arithmetic to hold in a semi-group is obtained from the theory of residuated lattices and a new inductive proof of the theorem is given for the lattice of positive integers. (Received October 30, 1940.)

13. Hermann Weyl: *Theory of reduction for arithmetical equivalence.* II.

Instead of the arithmetically refined method of reduction used in the first paper, the author now resorts to a rougher method which also goes back to Minkowski, and works without the assumption that the class number of ideals is 1. Its generalization to algebraic number fields F with several infinite prime spots is due to Siegel and P. Humbert (Commentarii Mathematici Helvetici, vol. 12 (1939–1940), pp. 263–306). The author follows the same geometric approach as before, including all classes of lattices over F and adding to the case of a field F that of a field quaternion algebra with totally positive norm over a totally real field. (Received November 23, 1940.)

Analysis

14. Warren Ambrose: *Representation of ergodic flows.* Preliminary report.

A flow is a one-parameter group T_t $(-\infty < t < \infty)$ of measure preserving transformations of a space Ω into itself. It is measurable if the function T_tP is a measurable function on $T \times \Omega$, where T denotes the real line taken with Lebesgue measure. For a measurable ergodic flow it is shown that a necessary and sufficient condition that a group of unitary operators U_t defined by $U_tf(P) = f(T_tP)$ have an eigenvalue (other than the trivial eigenvalue 1) is that the flow be isomorphic (with respect to measure properties) to a flow built on a measure preserving transformation (for definition of such a flow see abstract 46-11-446), thus showing that if an ergodic flow has an eigenvalue the measure on Ω must be the direct product measure of a cross section measure with Lebesgue measure along the trajectories of the flow. It is also shown that if the flow has no eigenvalues but satisfies certain conditions which are stronger than

measurability then the measure on Ω is such a direct product measure. (Received November 18, 1940.)

15. R. P. Boas and D. V. Widder: *Completely convex functions.* Preliminary report.

A function $f(x)$ is defined to be completely convex in an interval (a, b) if it has even derivatives which satisfy the relation $(-1)^n f^{(2n)}(x) \geqq 0$ there. For example, the functions $\sin x$ and $\cos x$ have the property in the intervals $(0, \pi)$ and $(-\pi/2, \pi/2)$ respectively. It was shown by Widder (Proceedings of the National Academy of Sciences, November, 1940) that such a function must be entire of order at most unity (and type at most $\pi/(b-a)$ if the order is one). A simplified proof of this result is given, and an integral representation for completely convex functions is obtained. Inversion formulas for the integral involved in this representation are discussed. (Received November 23, 1940.)

16. Russell Cowan: *A method of solving a linear difference equation with polynomial coefficients of degree m.*

Given a linear difference equation Δ of order p with polynomial coefficients of degree m, a linear differential equation D is found whose recursion relation yields Δ. The order of D is the smaller of p and m. Under slight restrictions on the coefficients of Δ, the differential equation has $p+2$ regular singular points. The case $p=2$ is studied in detail. By transforming D to Heun's equation, the solution of Δ is readily obtained. In special cases one of the singular points of D is not regular. But even under these circumstances, Δ can be solved by utilizing the characteristic exponent λ which is found by transforming D into a Riccati equation. (Received November 18, 1940.)

17. J. H. Curtiss: *Degree of polynomial approximation on a lemniscate.*

It is known that a function $F(z)$ analytic and single-valued interior to the lemniscate Γ: $|\omega(z)| = \mu$, $\omega(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_\lambda)$, can be expanded in a series of the form $\sum_0^\infty q_r(z)[\omega(z)]^r$ convergent to $F(z)$ for $|\omega(z)| < \mu$, where the functions $q_r(z)$ are polynomials of degree less than λ. Let $S_n(z; F)$ denote the nth partial sum of this series, let $S_n^{(r)}(z; F)$ denote the nth Cesàro mean of order r, $0 < r \leqq 1$, and let $J_n(z; F)$ denote the nth Jackson mean. The following theorem is proved: If C consists of one or several of the closed contours of Γ, and if $G(z)$ is a function analytic interior to C, continuous in the closed region or regions, and satisfying a Lipschitz condition of order η on C, $0 < \eta \leqq 1$, then there exists a function $F(z)$ analytic interior to Γ such that, uniformly on and interior to C, we have $S_n(z; F) - G(z) = O(n^{-\eta/m} \log n)$, $S_n^{(r)}(z; F) - G(z) = O(n^{-r}) + O(n^{-\eta/m})$, $r \neq \eta/m$, $= O(n^{-r} \log n)$, $r = \eta/m$, $J_n(z; F) - G(z) = O(n^{-\eta/m})$, where m is the order of the multiple point on C of highest order. If $C = \Gamma$, then $F(z) \equiv G(z)$. Examples are given to show that these estimates cannot be improved. (Received November 27, 1940.)

18. J. H. Curtiss: *Trigonometric interpolation by means of the complex Lagrange polynomial.*

Let $L_n(\theta; F)$ be the unique polynomial in $e^{i\theta}$ of degree at most $n-1$ which coincides with a given function $F(\theta)$ in the points $\theta = 2\pi k/n$, $k = 1, 2, \cdots, n$. Trivial examples show that even if $F(\theta)$ is analytic for all θ, the sequence $\{L_n(\theta; F)\}$ may diverge for

all $\theta \neq 0$, mod 2π. This paper studies the convergence of $\{L_n(\theta; F)\}$ for functions $F(\theta)$ which are continuous and are the boundary value functions of functions $f(z)$, $z = re^{i\theta}$, analytic and of class H for $|z| < 1$. The following results are typical: (a) If the Fourier series for $F(\theta)$ converges absolutely for all θ, then $L_n(\theta) \to F(\theta)$ uniformly; (b) if $F(\theta)$ is of bounded variation in an open neighborhood of θ_1, then $L_n(\theta) \to F(\theta)$ uniformly in a closed subneighborhood of θ_1; (c) $L_n(\theta; G_n) \to F(\theta)$ uniformly, where $G_n(\theta) = (n/2\pi) \int_\theta^{\theta+2\pi/n} F(t)dt$; (d) $L_n(\theta; G_n) - F(\theta) = O(n^{-\alpha})$ if $F(\theta)$ satisfies a Lipschitz condition with exponent $\alpha < 1$. These results are then generalized to the case of interpolation on a Jordan curve C in the manner of the author's paper in the Transactions of this Society, vol. 38 (1935), p. 458. (Received November 27, 1940.)

19. L. L. Dines: *On the mapping of quadratic forms.*

If $P(z)$ and $Q(z)$ are real quadratic forms in n variables $z_1, z_2, \cdots, z_n$, the transformation $x = P(z)$, $y = Q(z)$ maps the n-dimensional z-space into a set of points $\mathfrak{M}$ of the xy-plane. This paper obtains properties of the map $\mathfrak{M}$, and from them deduces certain results relative to the pair of forms $P(z)$ and $Q(z)$. Among the results is the theorem proved by Albert (this Bulletin, vol. 44 (1938), p. 250) and by Reid (this Bulletin, vol. 44 (1938), p. 437). The paper will appear in the Bulletin. (Received November 14, 1940.)

20. Jesse Douglas: *Solution of the inverse problem of the calculus of variations.*

A solution is presented of the classical problem: given a family of curves in 3-space, as defined by differential equations $y'' = F(x, y, z, y', z')$, $z'' = G(x, y, z, y', z')$; determine the existence or nonexistence of a calculus of variations problem of the form $\int \phi(x, y, z, y', z')dx = \text{minimum}$ having the prescribed curve family for the totality of its extremals, and, in the affirmative case, to find the most general form of the function ϕ. Preliminary notes stating the results and describing the methods used have already appeared in the Proceedings of the National Academy of Sciences, vol. 25 (1939), pp. 631–637, and vol. 26 (1940), pp. 215–221. A detailed paper will be published in the Transactions of this Society. (Received November 20, 1940.)

21. J. J. Eachus: *Classification of solutions and of pairs of solutions of $y''' + 2p(x)y' + p'(x)y = 0$ by means of boundary conditions.*

G. D. Birkhoff (Annals of Mathematics, (2), vol. 12 (1911), p. 103) has shown that the differential equation $y''' + 2p(x)y' + p'(x)y = 0$ has three types of solutions according to the number and nature of zeros of the solution. He has further shown that there are only a limited number of possible configurations of the zeros of two solutions of the equation. These same conclusions are deduced in this paper in a new manner, one which leads to a method for determining the "type" of a solution y_1 from the values of y_1 and its first two derivatives at any point x_0, and a method for determining the configuration of the zeros of y_1 and y_2 from the values of y_1 and y_2 and their first two derivatives at any point x_0. (Received November 25, 1940.)

22. Samuel Eilenberg: *Linear measure and convexity.*

Let X be a continuum with the metric ρ. If the linear measure $L(X, \rho)$ is finite then there is a metric ρ' equivalent with ρ and such that (1) ρ' is convex (2) $\rho' \geqq \rho$ (3) $L(X, \rho') = L(X, \rho)$. The metric ρ' is unique. (Received October 24, 1940.)

23. K. W. Folley: *A property of a simply ordered set.*

It is shown in this paper that a method of proof which was employed by Sierpiński to prove a theorem concerning the decomposition of real numbers may be used to prove an analogous theorem concerning the decomposition of a simply ordered set which contains an everywhere dense η_α subset. This result leads to a proposition which is proved to be equivalent to the generalized hypothesis of the continuum. (Received October 25, 1940.)

24. H. L. Garabedian: *Relations between hypergeometric methods of summation.*

The definition of hypergeometric summability $(H, \alpha, \beta, \gamma)$ is extended and new inclusion and equivalence relations between hypergeometric methods of summation are obtained. The relation $(H, \alpha_1, 1, \alpha_1+\gamma) \approx (H, \alpha_2, 1, \alpha_2+\gamma)$, $\Re(\alpha_1, \alpha_2, \gamma)>0$, is probably the most significant result established. Corollaries to this result are $(H, \alpha, 1, \alpha+\beta) \approx (C, \beta)$, $\Re(\alpha, \beta)>0$; $(H, \alpha, 1, \gamma) \approx (C, \gamma-\alpha)$, $\Re(\alpha, \gamma, \gamma-\alpha)>0$. These statements imply that, given any hypergeometric method of summation of the type $(H, \alpha, 1, \gamma)$, there exists a Cesàro method of summation equivalent to it; and, conversely, given any Cesàro method of summation (C, β) there exists a single infinitude of hypergeometric methods of summation of the type $(H, \alpha, 1, \alpha+\beta)$ equivalent to it. (Received November 20, 1940.)

25. J. W. Green: *A special type of conformal map.*

Let G_z be the unit circle in the z plane, O an open set of points on the circumference of G_z, and F its complement with respect to the circumference. The set F is supposed to be of positive measure. In the case $\bar{O}-O$ is of zero measure, there exists a function $w=w(z)$ which maps G_z conformally on the interior of the unit circle G_w with certain radial slits deleted. This transformation carries an interval of O into all or part of a radial slit, and carries points of F into points of $|w|=1$, almost everywhere on F. (Received November 25, 1940.)

26. R. E. Greenwood: *Hankel and other extensions of Dirichlet series.*

This paper considers series of the form $g(s)=\sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) G(\lambda_n s)$. If $G(z)\equiv 1$, then the series reduces to the well known Dirichlet series. If $G(z)$ is analytic in the right half-plane, is bounded in the half-plane to the right of $z_0>0$ and has a certain asymptotic expansion, the series above will be convergent in a half-plane, and also absolutely convergent in a half-plane. The abscissae of convergence are the same as the abscissae for the corresponding Dirichlet series. Summation formulae involving the first m coefficients are developed, and these reduce to Perron's formula for Dirichlet's series when $G(z)\equiv 1$. As a particular case such a function $G(z)$ is chosen so that $g(s)$ is a series of Hankel functions of the first kind of fixed order ν on the imaginary axis, and a few special results are obtained for this series. (Received November 14, 1940.)

27. H. J. Hamilton: *On monotonic and convex solutions of certain difference equations.*

The principal result of this paper is reduction of the problems of existence and of uniqueness (excepting an additive constant) of continuous, convex solutions U of the equation $U(x+1)-U(x)=G(x)$ to the same problems for monotone non-decreasing

solutions of a related equation of the same form. This reduction extends itself partially to linear difference equations of higher order with constant coefficients. Our conclusions are applied to some results of Fritz John. (Received September 28, 1940.)

28. L. B. Hedge: *Transformations of multiple Fourier series.* Preliminary report.

If $(n)=(n_1, n_2, \cdots, n_k)$ is a lattice point of euclidean k-space and $(x)=(x_1, x_2, \cdots, x_k)$ is a point of euclidean k-space, and if $a_{(n)}$ and $\lambda_{(n)}$ are functions on the lattice points of k-space, we consider the series $S=\sum a_{(n)}e^{i(n\cdot x)}$ and its transform $\lambda S=\sum\lambda_{(n)}a_{(n)}e^{i(n\cdot x)}$ where $(n\cdot x)=n_1x_1+n_2x_2+\cdots+n_kx_k$. A classification, analogous to that used for single series, is defined for the series S and λS, and transformations λ which take one class into another are characterized for interesting cases. The results include and extend the known transformation theory for single trigonometric series. The spherical summation of Bochner and a recent moment problem solution by the author are the principal tools used for the study. (Received December 2, 1940.)

29. A. E. Heins: *On the solution of partial difference equations. Two interval "boundary conditions."* Preliminary report.

The solution of the partial difference equation $f(x+1, t)+f(x-1, t)=2f(x, t+1)$ is considered under the following initial and boundary conditions: For $[t]=0$, $f(x, t)$ is prescribed, $af(x-[x], t)+f(x-[x]+1, t)=0$ and $bf(x-[x]+p, t)+f(x-[x]+p+1, t)=0$. The partial difference equation is first reduced to an ordinary difference equation with the Laplace transform and the resulting ordinary difference equation solved under the given boundary conditions. The solution depends on a finite expansion of sines and cosines of multiples of angles which are roots of a certain transcendental equation. The nature of this rather interesting expansion will be discussed at a later date. (Received November 18, 1940.)

30. M. R. Hestenes: *Extension of the range of a differentiable function.*

In the present paper two methods of extending the range of a differentiable function are given. The first method is essentially a reflection across boundaries. By this method it is shown that a function $f(x_1, \cdots, x_n)$ of class C^m (m finite) on a closed set A with a suitable boundary can be extended to be of class C^m over the whole of euclidean n-space. The second method is similar to one used by Whitney (Transactions of this Society, vol. 36 (1934), pp. 63–89) to show that a function of class C^m on an arbitrary closed set A can be extended to be of class C^m over the whole space and to be of class C^∞ at the points not in A. (Received November 26, 1940.)

31. Einar Hille: *A class of differential operators of infinite order.* II.

Let $D_zw=P_0(z)w''+P_1(z)w'+P_2(z)w$. Let $G(z)$ be an entire function of order $\frac{1}{2}$ and minimal type. Then the operator $G(D_z)$ preserves holomorphism in any domain where the P's are holomorphic. The condition on $G(z)$ is also necessary for most operators D_z of practical importance. The equation $G(D_z)\cdot U=0$ can be studied by the methods of Ritt and Valiron. If the P's are single-valued and $a_1, \cdots, a_m$ are the singularities of $D_zw=0$, let R be an unbounded non-ramified covering surface of the plane punctured at $a_1, \cdots, a_m$ on which all solutions of $(D_z-\lambda_n)w=0$ are single-valued, where $\{\lambda_n\}$ are the roots of $G(z)=0$. $U(z)$ is single-valued on R. Its domain of existence E is such that any maximal component of $R-E$ has at least one point a_k

as limit-point. If R is simply-connected, so is E. $U(z)$ has a unique Ritt series in terms of solutions of $(D_z-\lambda_n)w=0$. If $P_0=P_1=1$, P_2 entire, and the λ_n suitably restricted, this series converges in E, and E is convex. The results extend to differential operators D_z of order n. (Received October 5, 1940.)

32. Fritz John: *On the character of solutions of hyperbolic equations.* Preliminary report.

Let $u(x_1, \cdots, x_n)$ be a solution of a linear, normal-hyperbolic differential equation of second order with analytic coefficients. The solution u is shown to be "pseudo-analytic" in the following sense: Given a family of "time-like" analytic curves C_α depending on a parameter α, all C_α having the same two endpoints, then $\int u dt$ is an analytic function of α, t being an analytic parameter on C_α. It follows that the values of u on any time-like two-dimensional analytic manifold, interior to the domain of regularity of u, are not independent. This is a generalization of results previously obtained by the author for a special equation (Mathematische Annalen, vol. 111, pp. 541–555). The proof makes use of the form of the solution obtained by J. Hadamard in his *Lectures on Cauchy's Problem.* (Received November 20, 1940.)

33. J. P. LaSalle: *A note on pseudo-normed linear spaces.*

The equivalence of speudo-normed linear spaces (p.l.s.'s) and linear topological spaces (l.t.s.'s) has been shown by Hyers. (See Duke Mathematical Journal, vol. 5 (1939), pp. 628–634.) In the present paper l.t.s.'s are studied as characterized in terms of the pseudo-norm. It is shown that the existence of an open convex set containing the zero element and properly contained in a p.l.s. T is a necessary and sufficient condition for the existence of a non-null linear functional on T. An example is given of a p.l.s. on which no non-null linear functional can be defined. Also the set of all linear functions on a p.l.s. T to a p.l.s. T' is shown to be a pseudo-normed linear space. If T' is convex, then the p.l.s. of all linear functions on T to T' is itself convex. Hence the set of all linear functionals on a p.l.s. T is a convex p.l.s. If a p.l.s. T is locally bounded, then the set of all linear functionals on T is shown to be a normed linear space. (Received October 22, 1940.)

34. J. P. LaSalle: *Pseudo-normed linear sets over valued rings.* I.

In this paper a generalization of linear spaces is given by replacing real number multipliers by multipliers taken from a valued ring. A valued ring is defined to be a ring A with a unity element such that there is defined on A a real-valued function $M(\alpha)$ where (1) $M(\alpha)\geqq 0$ for all $\alpha\in A$; (2) $M(\alpha\beta)\leqq M(\alpha)M(\beta)$; (3) $M(\alpha+\beta)\leqq M(\alpha)+M(\beta)$; (4) $M(-1)=1$; (5) $M(\alpha)>1$ for some $\alpha\in A$. The set of all formal polynomials with coefficients taken from a ring with a unity element can be shown to be a valued ring, which indicates somewhat the generality of valued rings. A linear set T over a valued ring A is said to be pseudo-normed w.r.t. a strongly partially ordered set D if there exists a real-valued function $n(x, d)$ on TD such that (1) $n(x, d)\geqq 0$; $n(x, d)\leqq 1$ for all $x\in T$ implies $x=\Theta$; (2) $n(\alpha x, d)\leqq M(\alpha)n(x, d)$; (3) given $d\in D$ there exist $e\in D$ such that $n(x, e)\leqq 1$, $n(y, e)\leqq 1$ implies that $n(x+y, d)\leqq 1$; (4) given $x\in T$ and $d\in D$ there exist $\alpha\in A$ and $y\in T$ such that $x=\alpha y$, $n(y, d)\leqq 1$; (5) given $\alpha\in A$ and $d\in D$ there exist $e\in D$ such that $n(x, e)\leqq 1$ implies that $n(\beta x, d)\leqq 1$ for all $M(\beta)\leqq M(\alpha)$; (6) $e\geqq d$ implies that $n(x, e)\geqq n(x, d)$. Another characterization of T is given in terms of a neighborhood system. The topological properties of such a space T are then studied. Later a generalized concept of linear functions and differentials will be introduced for such spaces. (Received October 22, 1940.)

35. Walter Leighton: *On the convergence of continued fractions.*

Let $z_2, z_3, \cdots$ be any bounded sequence of complex numbers such that $R(z_n) \geqq \epsilon > 0$ $(n=2, 3, \cdots)$, where ϵ may be taken arbitrarily small. The continued fraction $1+K_1^\infty [a_n/1]$ converges, where $a_n=(z_{n-1}+1/2)(z_n+1/2)$ $(n=2, 3, \cdots; z_1=1)$. (Received November 18, 1940.)

36. J. Marcinkiewicz and Antoni Zygmund: *On the behavior of trigonometric and power series.*

A point set Z on the plane is said to be of circular structure with center z_0 if whenever a point ζ belongs to Z, so does the whole circle $|z-z_0| = |\zeta - z_0|$. Given a power series $\sum_0^\infty c_n z^n$, let $\sigma_n^\alpha(\theta)$ denote the αth Cesàro means of the series $\sum_0^\infty c_n e^{in\theta}$ and let $L^\alpha(\theta)$ denote the set of the limit points of the sequence $\sigma_0^\alpha(\theta), \sigma_1^\alpha(\theta), \cdots, \sigma_n^\alpha(\theta), \cdots$. The chief result of the paper may be stated as follows: If for every θ belonging to a set E of positive measure the series $\sum_0^\infty c_n e^{in\theta}$ is summable $(C, \alpha+1)$ $(\alpha > -1)$ to sum $t(\theta)$, then for almost every θ of E the set $L^\alpha(\theta)$ is of circular structure with center $t(\theta)$. (Received October 31, 1940.)

37. R. S. Martin: *Minimal positive harmonic functions.*

The idea of a minimal positive harmonic function arises as follows. The Poisson-Stieltjes integral formula for a sphere depends upon (1) a family of positive harmonic functions (namely the kernel $F(S, P)$ where the point S lies on the boundary and is considered as a parameter), and (2) a family of linear operations (namely integration with respect to mass-distributions on the boundary). Given then a general bounded domain D, one may ask for suitable generalizations of the situations just described. The idea of the present paper is to consider, as a generalization of the family $F(S, P)$ corresponding to the sphere, the family of those functions which are *minimal*, positive and harmonic in D. A function $u(P)$, positive and harmonic in D, is called *minimal* for D if it dominates there no positive harmonic functions, except those of the form $cu(P)$, where $c>0$ is a constant. The purpose of the paper is to discuss the theory of representations of harmonic functions based on this idea. (Received October 22, 1940, from Tibor Radó.)

38. C. N. Moore: *On the Cesàro and Abel-Poisson summability of the differentiated double Fourier series.*

It has been shown that at points where $f(x)$ has a derivative or generalized derivative the differentiated Fourier series will be summable (C, r) for $r>1$ or will be summable by the Abel-Poisson method. In the present paper these results are extended to the partial derivatives or generalized partial derivatives of functions of two variables and the corresponding differentiated double Fourier series. The result concerning Abel-Poisson summability appears as a corollary of the result concerning Cesàro summability by use of general convergence factor theorems. (Received November 27, 1940.)

39. G. D. Nichols: *A sufficient condition for Cesàro summability.*

The series $\sum f(r) \cos rx$ and $\sum f(r) \sin rx$ are shown to be summable (C, k) provided the $\Delta^k f(r)$ is a monotone null sequence. In case $f(r)$ is a polynomial of degree $k-1$, a closed expression is obtained for the Cesàro sum. (Received November 23, 1940.)

40. I. E. Perlin: *Indefinitely differentiable functions of several real variables.*

In the present paper the author considers indefinitely differentiable functions of several real variables. Let $M_{ij\ldots k}$=l.u.b. $\left|\partial^{i+j+\cdots+k}f(x_1, x_2, \cdots, x_n)/\partial x_1^i \partial x_2^j \cdots \partial x_n^k\right|$ on $0 \leq x_p \leq a_p$. Sufficient conditions that there exist a function $f(x_1, x_2, \cdots, x_n)$ with prescribed $M_{ij\ldots k}$ are established. (Received November 25, 1940.)

41. Arthur Sard: *The measure of the critical values of differentiable maps of euclidean spaces.*

Consider the map $y^j = f^j(x^1, \cdots, x^m)$, $j=1, 2, \cdots, n$, of a region R of euclidean m-space into part of euclidean n-space. Suppose each of the functions f^j is of class at least C^q in R ($q \geq 1$). A critical point of the map is a point in R at which the matrix of first derivatives, (f_i^j), $i=1, 2, \cdots, m$; $j=1, 2, \cdots, n$, is of less than maximum rank. A critical value y is the image $f(x)$ of a critical point x. The following theorem is proved: If $m \leq n$, the set of all critical values is of m-dimensional measure zero (in the sense of Hausdorff-Saks) without further hypothesis on q; if $m > n$, the set of all critical values is of n-dimensional measure zero providing $q \geq m-n+1$. For $n=1$, this result specializes to a known theorem, surmised by Marston Morse and proved for all m by A. P. Morse (Annals of Mathematics, (2), vol. 40 (1939), pp. 62–70). The proof of the theorem of the present paper depends in part on a result of A. P. Morse. The hypothesis on q cannot be weakened, as can be seen from an example due to H. Whitney (Duke Mathematical Journal, vol. 1 (1935), pp. 514–517). (Received November 25, 1940.)

42. H. M. Schwartz: *Convex functions and the law of the mean.*

Let S denote the class of functions of a real variable for which the law of the mean holds throughout a given interval, and let T denote the class of functions which are either strongly convex or strictly concave in that interval. The classical conditions insuring that the function f be in S are for f of T necessary as well as sufficient. The class U common to both T and S is uniquely determined by the requirement that the θ-function occurring in the law of the mean be a single-valued function of its arguments. This θ-function for f of U is studied both as to its dependence on f and as to its dependence on the variable and its increment. It is shown, for instance, that if $f_n \in S$ ($n=1, 2, \cdots$), sequence f_n' converges continuously to f' in the interval, and $f \in U$, then $\theta(f_n)$ converges to $\theta(f)$ in the domain of its arguments. Most of the results are extended to functions of many variables. (Received November 20, 1940.)

43. H. M. Schwartz: *Sequences of Stieltjes integrals.*

The paper contains a study of some of the properties of function sequences of the form $\int_a^x f dg_n = s_n$ ($n=1, 2, \cdots$), $a < x \leq b$ (a, b finite or infinite), where f, g_n are bounded functions for which the integrals exist in the sense of Riemann-Stieltjes. When g_n are of bounded variation an obvious sufficiency condition for the convergence of s_n in (a, b) can be stated in terms of the integrability conditions of f with respect to g_n, but it is of interest to find such conditions in terms of the convergence properties of the sequence g_n. This problem is solved here only partially. Assuming that g_n are uniformly of bounded variation in (a, b), the convergence of s_n is proved for special classes of f and ordinary convergence of the sequence g_n, and generally for special modes of convergence of this sequence. A typical result is as follows: s_n converges in

(a, b) (assumed finite) if $g_n \to g_0$ and if $\int f dg_n$ $(n=0, 1, 2, \cdots)$, $\int f dv$ exist, where v is any of the limit functions of the sequence v_n, $v_n(x)$ being the total variation of g_n in (a, x). (Received November 25, 1940.)

44. Hyman Serbin: *Upper bounds for the remainder of certain power series.*

This note exemplifies a method of obtaining an upper bound of the remainders of power series in which the coefficients are connected by a recurrence relation with a constant number of terms. The function considered is wave mechanical $\exp(-ix) \cdot F(m+1-ai; 2m+2; 2ix)$ where $F(\alpha; \gamma; z)$ is the confluent hypergeometric function. A majorant of the remainder, suggested by the recurrence relation, is expressed in closed form. The upper bound so obtained is not explicitly given in terms of the parameters but depends on two successive coefficients. However, for the purpose of computation, the result is satisfactory. The above results were obtained in the course of work done by the Project for the Computation of Mathematical Tables, conducted by the Work Projects Administration of New York City. (Received October 30, 1940.)

45. W. S. Snyder: *On independence of the path for line integrals of continuous functions.*

Let $p(x, y)$ and $q(x, y)$ be defined and continuous in an oriented rectangle R of the xy-plane. A necessary and sufficient condition that $\int p dx + q dy$ be independent of the path in R is derived in terms of the double integrals of the functions p and q. The author also formulates and answers a more general form of a question raised by Menger (Proceedings of the National Academy of Sciences, vol. 25 (1939), p. 623), showing that the conditions of Menger and Fubini (Proceedings of the National Academy of Sciences, vol. 26 (1940), p. 199) cannot be satisfied for "undotted" nets. (Received November 23, 1940.)

46. Abraham Spitzbart: *Approximation in the sense of least pth powers by polynomials with a single auxiliary condition of interpolation.*

Let $F_0(z)$ be the function, analytic interior to the analytic Jordan curve C, of class E_p $(p>0)$ in that region, and satisfying the condition $F_0(\alpha)=A$ ($z=\alpha$ a point interior to C, A an arbitrary contant), which minimizes the integral $\int_C |F(z)|^p |dz|$. Let $P_n(z)$ be the minimizing polynomial of degree n for the integral $\int_C |P_n(z)|^p |dz|$, $P_n(\alpha)=A$. It is proved that the sequence $P_n(z)$, $n=0, 1, 2, \cdots$, converges maximally to the function $F_0(z)$ on the closed set Γ consisting of C plus its interior (for definition see J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, American Mathematical Society Colloquium Publications). The results extend to the case where a suitable norm function $N(z)$ is introduced into the integral to be minimized. (Received November 25, 1940.)

47. Otto Szász: *On the partial sums of harmonic developments and related power series.*

Let T denote the class of all harmonic sine developments $H(r, \theta) = \sum C_\nu r^\nu \sin \nu\theta$, convergent for $0<r<1$, and non-negative for $0<\theta<\pi$. There is a largest R^* such that for every H in T, $\sum_1^n C_\nu r^\nu \sin \nu\theta$ is non-negative for $0<r<R_n$, $0<\theta<\pi$. Then R_n is characterized algebraically, which yields an asymptotic estimate. The result is

applied to Fourier series of convex functions and to related power series. (Received November 26, 1940.)

48. A. E. Taylor: *Conjugations of complex Banach spaces.*

Let E be a complex Banach space. An operation T on E to E is called a conjugation if it is additive, and if $Taz=\bar{a}Tz$, $T^2z=z$. A set M in E is called an essentially real linear manifold (ess. r.l.m.) if it is closed under addition, and under multiplication by real numbers, and if x, ix in M implies $x=0$. An ess. r.l.m. is maximal if it is not a proper subset of any ess. r.l.m. If M is a maximal ess. r.l.m. every element of E is uniquely representable in the form $z=x+iy$, where x, y are in M. The operation $T(x+iy)=x-iy$ is a conjugation. T is continuous if and only if M is closed. Also, if T is any conjugation, the set of elements x such that $Tx=x$ is a maximal ess. r.l.m. It may be shown that every E has a maximal ess. r.l.m. The author does not know if there always exists a closed, maximal ess. r.l.m., but when one does exist it is possible to re-norm the space, without altering the topology, in such a way that $\|x+iy\| = \|x-iy\|$. (Received October 26, 1940.)

49. J. M. Thomas: *Orderly differential systems.*

Riquier's existence theorem for orthonomic systems is extended to a larger class of systems called orderly. Substantially the defining property of an orderly system is that it can be decomposed into orthonomic components. The proof of the existence theorem given is new, even for the case of orthonomic systems. Its chief feature is the elimination of the integers called cotes by Riquier. They are replaced by two systems of inequalities which it is thought arise more naturally and have simpler nature than the systems of inequalities among the cotes. (Received November 20, 1940.)

50. W. J. Trjitzinsky: *Analytic theory of singular elliptic partial differential equations.*

This work presents an extensive analytic theory of the equation $F(u)\equiv\sum a_{i,k} \cdot \partial^2 u/\partial x_i \partial x_k + \sum b_i \partial u/\partial x_i + (c+\lambda)u = f$ $(i, k=1, 2, \cdots, m)$, where λ is a parameter and the coefficients are functions of $(x_1, \cdots, x_m)$ continuous in an open domain D, the first partials of the b_i and a number of partials of the a_i, k are continuous in D, and the functions, involved, may become infinite near the frontier of D. The investigation was suggested by certain analytic aspects of Schrödinger wave equations. The equation is transformed into an integral equation with the aid of a suitable function of the geodesic distance (formed for a suitable line element); the integral equation plays an essential role in the work, in particular, Fredholm's theory is locally applicable in D. Associated with F there is constructed a sequence of "regular" approximating homogeneous boundary value problems, (P_1), (P_2), $\cdots$. When F is self adjoint, theorems on existence of solutions, uniqueness properties and so forth are given for λ nonreal; also, for λ real, depending on sufficient "rarefication" of the totality of characteristic values formed for (P_1), (P_2), $\cdots$. An essential role is also played by an appropriate spectral theory based on (P_1), (P_2), $\cdots$. (Received November 28, 1940.)

51. H. S. Wall: *A theorem on real functions bounded in the unit circle.*

Let E denote the class of functions $e(x)$ which are analytic for $|x|<1$, real when x is real, and for which $M(e)=\text{l.u.b.}_{|x|<1}|e(x)|\leqq 1$. Let F denote the subclass of all

functions $f(x)$ of E which are of the form $f(x)=\int_0^1 d\phi(u)/(1+xu)$ where $\phi(u)$ is real, bounded and monotone nondecreasing on the interval $0\leqq u\leqq 1$. Then there is a one-to-one correspondence between the functions of E and of F such that if $e(x)\leftrightarrow f(x)$, then $e(x)=[1-x-2f(z)]/[1-x+2xf(z)]$, $z=4x/(1-x)^2$, $|x|<1$. (Received November 7, 1940.)

52. J. L. Walsh and E. N. Nilson: *Approximation to an analytic function by functions analytic and bounded in a region.*

Let R be a finite sum of disjoint regions containing a closed set S in its interior, with the respective boundaries C_1 and C_0 composed of a finite number of analytic Jordan curves, each component region of R containing at least one point of S, no point $R-S$ separated by S from C_1. Define $u(z)$ as unity on C_1, zero on C_0, continuous in the extended plane, harmonic except on C_1 and C_0. If $v(z)$ is conjugate to $u(z)$ in $R-S$, then $u(z)=u(\infty)-(1/2\pi)\int_{C_1+C_0}\log|z-\zeta|\,dv(\zeta)$, for z not on C_1+C_0. This equation is used to define poles and points of interpolation for sequences of rational functions which are used (Walsh, Proceedings of the National Academy of Sciences, vol. 24 (1938)) to prove: Let $f(z)$ be analytic on S but not throughout R (a single region); for $M>0$, let $f_M(z)$ be that function analytic and of modulus not greater than M in R for which $[\max|f(z)-f_M(z)|,\ z \text{ on } S]$ is least. Then, for $0\leqq\sigma<\rho$, $\limsup_{M\to\infty}[\max|f(z)-f_M(z)|, z \text{ in } \overline{R}_\sigma]^{1/\log M}=e^{(\sigma-\rho)/(1-\rho)}$, where R_σ is the set on which $0\leqq u(z)<\sigma$ and ρ is the largest number, $0<\rho<1$, such that $f(z)$ is analytic throughout R_ρ. (Received December 2, 1940.)

53. W. F. Whitmore: *Convergence theorems for functions of two complex variables.* II.

In continuation of the work of the first part of this paper (American Journal of Mathematics, vol. 62 (1940), pp. 687–696), results are obtained for convergence of a function of two complex variables in sector-domains $I^4=\sum_{|z_2|\leqq 1} I^2[\theta_2(z_2),\ \theta_1(z_2)]$ where $I^2(\theta_2,\theta_1)=E[\theta_2\geqq\arg z_1\geqq\theta_1]$ (superscript indices give the dimensionality of the domain in question). The limit function approached at the vertex surface $H^2=E[z_1=0, |z_2|\leqq 1]$ is also examined, and shown to be an analytic function of z_2. The principal tool is the theory of harmonic measure. These sextor-domains are useful as domains of comparison for arbitrary domains whose boundary hypersurfaces contain a segment of an analytic hypersurface. (Received October 28, 1940.)

54. D. V. Widder: *Necessary and sufficient conditions for the representation of a function in Lidstone series.*

A set of polynomials $\Lambda_n(x)$ is defined as follows: The first, $\Lambda_0(x)$, is x. For each positive integer n the second derivative of $\Delta_n(x)$ is to be $\Lambda_{n-1}(x)$, and each polynomial except the first is to vanish at zero and at one. For example, $\Lambda_1(x)$ is $(x^3-x)/6$. In terms of these polynomials a Lidstone series for the functions $f(x)$ has the form $f(x)=\sum_{k=0}^{\infty} f^{(2k)}(1)\Lambda_k(x)+f^{(2k)}(0)\Lambda_k(1-x)$. Any partial sum of the series is a polynomial whose coefficients are determined so that it may coincide together with as many of its even derivatives as possible with $f(x)$ and its corresponding even derivatives at zero and one. These series, discovered by G. J. Lidstone, have been studied by H. Poritsky, J. M. Whittaker and I. J. Schoenberg. (References may be found in Schoenberg's paper, this Bulletin, vol. 42 (1936), pp. 284–288.) In the present paper the series are considered from the real point of view. A new proof of a theorem of the author is given (Proceedings of the National Academy, November, 1940) that a

function $f(x)$ for which $(-1)^k f^{(2k)}(x) \geqq 0$ on an arbitrary interval is necessarily entire. By use of this result a necessary and sufficient condition that a function can be expanded in an absolutely convergent Lidstone series is obtained. (Received November 25, 1940.)

55. J. R. Woolson: *A Theory of projections in complex Banach spaces.* Preliminary report.

A projection P is defined as a linear operator on a complex Banach space B to B, such that $P^2=P$. Using an inter-space product $[F, f]$, $f \in B$, $F \in (B)$ the set of complex valued linear functionals defined on B, it is possible to prove the usual theorems concerning projections by certain analogies with the Hilbert space inner-product. (Cf. M. H. Stone, *Linear Transformations in Hilbert Space and Their Applications to Analysis*, American Mathematical Society Colloquium Publications, vol. 15.) The theory is used to show that $(1/n)\sum_{i=1}^{\infty} A^i$ converges weakly to a projection and to characterize a linear operator in terms of the linear manifolds which it leaves invariant. (Received October 28, 1940.)

Applied Mathematics

56. Harry Bateman: *Aerodynamical effects of changes in the fundamental equations.*

Simple examples indicate that a change from the elliptic to the parabolic type may not be as drastic in its effects as a change from the elliptic to hyperbolic type. The effects of the various changes that have been made in the equations of viscous flow by Prandtl, Oseen, Kármán and others are reviewed by the author. Some remarks are made also on mixtures of fluids and changes of state. Some new equations are considered and reference is made to the work of Duhem, Schutz, Silberstein, Natanson, Bjerknes and Kozlowski. Some remarks are made on the equations of Burgers, Mattioli and others which differ from the classical equations and yet exhibit some of the phenomena of turbulence. (Received November 18, 1940.)

57. M. A. Biot: *Finite difference equations applied to aircraft engine vibrations.*

The vibration amplitudes of the crankshaft are shown to satisfy a finite difference equation of the second order with constant coefficients. The frequency equation derived from the end conditions is solved numerically by an artifice which shortens considerably the time required for the evaluation of the critical speeds. (Received November 18, 1940.)

58. J. Bjerknes: *Some uses of mathematics in meteorology.*

The atmosphere exhibits tide-like wave phenomena, but the lunar component of the atmospheric tide is only about one tenth of the solar component. If the atmospheric tide is gravitational, just as the ocean tides, the atmosphere must be able to act as a vibrating system with a proper period very close to 12 solar hours and thereby give the solar component of the tide a greater amplitude than the lunar component. That vibration problem is mathematically rather well defined but still unsolved in its general form. The atmospheric disturbances responsible for the day to day variations of weather are mainly aperiodic and disorderly but at times they are quasi-periodic and to some extent amenable to mathematical treatment. The near-

est approximation to wave-like conditions are found in the upper atmosphere. The general westerly current which prevails there takes a sine curve shape under disturbed conditions and the sinuosities move along with the current but with less speed than the air. That wave phenomenon is mathematically less tractable than the tidal waves, mainly because of the difficult boundary conditions, but whatever little progress can be made with it would be a valuable contribution to the science of weather forecasting. (Received November 18, 1940.)

59. B. E. Gatewood: *Thermal stresses in regions bounded by epitrochords.*

Muschelisvili's method of solving the biharmonic equation when the first derivatives are known on the boundary of the region is used in solving the thermal stress problem for a long cylindrical body whose cross section is bounded by an epitrochord (see abstract 45-9-314.) (Received November 14, 1940.)

60. F. B. Hildebrand: *The approximate solution of singular integral equations arising in engineering practice.*

In this paper a method of numerical solution of integral equations given by Crout (Journal of Mathematics and Physics, M.I.T., vol. 19 (1940), pp. 34–92) is extended to the solution of certain integral equations wherein the unknown function, as well as the kernel, may be singular. Considering an equation of the first kind, $\phi(x) = \int_a^b \sigma(\xi) G(x, \xi) d\xi$, the unknown function $\sigma(x)$ is approximated by a linear combination $s(x)$ of suitably chosen singular functions, $s(x) = \sum_{i=1}^{r} \chi_i s_i(x)$, where the χ's are undetermined constants, plus a polynomial $y(x)$ of order $2n$, in the Lagrangean form $y(x) \sum_{i=-n}^{n} K_i(x) y_i$, where the y's are the undetermined ordinates at $2n+1$ equally spaced points in (a, b). The conditions $\phi(x_k) \approx \int_a^b \{s(\xi) + y(\xi)\} G(x_k, \xi) d\xi$, $k = 1, 2, \cdots, m$, where $m \geqq 2n+r+1$, constitute m linear equations in the $2n+r+1$ parameters, the parameters then being determined by a method of least squares. Several specific problems arising in static field theory and elasticity are solved numerically, the results being in good agreement with exact results obtained by other methods. The computation involved is organized by the use of matrices and reduced principally to operations for which the modern computing machine is well adapted. (Received November 25, 1940.)

61. Theodore von Kármán and W. R. Sears: *Solution of the problem of the airplane wing of finite span by expansion in eigen-functions.* Preliminary report.

The problem of the calculation of the lift distribution along the span of a wing according to the lifting-line theory of L. Prandtl can be reduced to the determination of a two-dimensional potential function which satisfies on the boundary of a circle a linear relation with variable coefficients between the values of the function and its normal derivative. In engineering practice this problem is usually solved by means of a development of the boundary values of the potential function in a trigonometric series. In the present paper the classical method of expansion in a series of the eigen-functions of the problem is carried out. It is shown that this method has several practical advantages; moreover, the eigen-functions and eigen-values required for the calculation are independent of the twist of the wing and each set of them is applicable for a certain family of planforms. They can be estimated by various approximate methods. (Received November 18, 1940.)

62. W. D. Rannie: *Tensor methods in the theory of turbulence.* Preliminary report.

Applications of tensor analysis to the statistical theory of isotropic turbulence as developed by T. von Kármán and L. Howarth, and later treated by H. P. Robertson, are reviewed. (Received November 18, 1940.)

63. Eric Reissner: *A new derivation of the equations for the deformation of elastic shells.*

The equations of the theory of small deformations of shells, first given by A. E. H. Love, are rederived in a simpler manner. The simplifications are accomplished by using (1) vector stress resultants and equilibrium conditions in vector form and (2) the three-dimensional system of orthogonal coordinates which goes with the lines of curvature on the middle surface of the shell and the strain components with respect to this system. The assumption that the normal to the undeformed middle surface is deformed into the normal to the deformed middle surface, satisfied by determining appropriate displacement components, is introduced into these strain components. (Received November 25, 1940.)

64. H. J. Stewart: *Steady state oscillations in an atmosphere on a rotation sphere.* Preliminary report.

If one plots the mean surface atmospheric pressure, averaged over a period of at least a week, one finds that in addition to the mean westerly flow of air, there exist large scale closed isobaric systems which change very slowly with time. Attempts to develop long range weather forecasting techniques have shown the positions of these systems to be of primary importance and a knowledge of the factors which control these systems is very useful as a guide in formulating forecasting methods. In the present paper certain steady state oscillations of the stratosphere are investigated and are shown to vary with the mean velocity in the same manner as the observed oscillations. (Received November 18, 1940.)

GEOMETRY

65. P. O. Bell: *On differential geometry intrinsically connected with a surface element of projective arc length.*

In this paper a surface element of projective arc length is interpreted geometrically and used to obtain a new geometric interpretation for each of the following: a generalization of Bompiani's projective curvature, a generalization of Fubini's asymptotic curvature, a projective torsion introduced in this paper, conjugate tangents, the tangents of Darboux, and the tangents of Segre. The associate conjugate net of an arbitrary net $N_{\lambda_1\lambda_2}$ of a surface S (introduced in this paper) is defined as the conjugate net whose tangents at a point P of S separate harmonically the tangents at P of the net $N_{\lambda_1\lambda_2}$. The following characteristic property of this net is a typical result: Let arcs PP_1, PP_2 of equal projective length s be measured, with respect to the form $ds=(2Rv')^{1/2}du$, from the point P along the curves C_{λ_1}, C_{λ_2}, respectively, of the net $N_{\lambda_1\lambda_2}$. The tangent plane to S at P intersects the line joining P_1P_2 in a point P_3 which tends to a limit point P_0, distinct from P, as s tends to zero. The tangent line joining PP_0 and its conjugate tangent envelop the conjugate associate of the net $N_{\lambda_1\lambda_2}$, as P varies over S. (Received November 20, 1940.)

66. L. M. Blumenthal: *Betweenness in metric ptolemaic spaces.*

A metric ptolemaic space (*MP*-space) is a metric space for which the determinant $|(p_ip_j)^2|$, $(i, j=1, 2, 3, 4)$, is nonpositive for every quadruple of points p_1, p_2, p_3, p_4 of the space. Investigating such spaces with regard to the four-, three-, and two-triple properties shows: (1) each *MP*-space has the four-triple property; (2) an *MP*-space has the three-triple property unless it contains the vertices of a convex tripod (that is, four points with one of them between each two of the remaining three points); (3) if four points of an *MP*-space contain exactly two linear triples, they may be labelled *a*, *b*, *c*, *d* so that either *abc* and *abd* or *acb* and *adb* hold. Thus an *MP*-space has the two-triple property unless it contains vertices of a fork or a bow. Besides the transitive property of the betweenness relation valid in all metric spaces, there is in *MP*-spaces the additional transitivity *abc* and *bad*$\rightarrow$*cad* and *abd*. These results are preliminary to a study of separable, complete, convex *MP*-spaces which contain for each three non-linear points at least one point equidistant from them. (Received November 23, 1940.)

67. L. M. Blumenthal and C. V. Robinson: *Helly theorems on the sphere.*

Theorems for families of convex subsets of the sphere are given corresponding to the following theorem due to E. Helly: If each $n+1$ members of a family of convex bodies of E_n intersect, then all the members of the family have a common point. On the sphere the existence of a common point is implied by the intersection of each 3, 4, 5, or 6 sets of the family according to the generality of the family considered. (Received November 26, 1940.)

68. C. R. Cassity: *The double points of a pencil of cubics invariant under the quadratic transformation.*

The double points of a general pencil of cubics contained in the invariant web of cubics of the involutorial quadratic transformation lie at the invariant points of the transformation and by pairs on the four lines which contain the six base points of the pencil which are not base points of the web. (Received November 25, 1940.)

69. Nathaniel Coburn: *A note on conformal geometry.*

Two Riemannian spaces of n-dimensions V_n and $'V_n$ are considered. If coordinate systems in $V_n(\xi^\lambda)$ and $'V_n(\xi^\lambda)$ can be chosen so that at corresponding points $P(\xi^\lambda)$, $'P(\xi^\lambda)$ the connections of the spaces are related by the conformal transformation, then are the spaces conformally related? The problem is shown to be equivalent to determining the number of independent solutions of a system of linear partial differential equations. By analyzing the integrability conditions of the system, it is shown that only one independent solution exists. Hence the spaces are necessarily conformal. The complete theorem is: If coordinate systems in $V_n(\xi^\lambda)$ and $'V_n(\xi^\lambda)$ exist so that at corresponding points $P(\xi^\lambda)$, $'P(\xi^\lambda)$ the connections of these spaces are related by the conformal transformation and if the principal directions of the metric tensor of $'V_n$ exist in V_n, then the spaces are conformal. This theorem is of interest in that it can be shown that the corresponding theorem is not valid in unitary spaces. (Received November 20, 1940.)

70. Nathaniel Coburn: *Unitary spaces with corresponding geodesics.*

In the first section of the paper, the equations of geodesic curves X_1, which depend

on a real parameter (t) and which are imbedded in a unitary space of n-dimensions K_n, are derived from the calculus of variations (Euler equations). The principal result is: the equations of such geodesics differ from the equations of geodesics in Riemannian space in that the former contain the torsion affinor whereas the latter do not contain this affinor. In the second section, the discussion centers on the connections of two unitary spaces K_n and $'K_n$ whose geodesics correspond. It is shown that: (1) if two unitary spaces, both with symmetric connections, have their geodesics in correspondence, then the connections are related by the projective transformations; (2) if K_n has a connection with torsion and $'K_n$ has a symmetric connection, then their geodesics cannot correspond. The first result is obtained in the same manner as the similar result in Riemannian space. The problem of determining all connections of unitary spaces K_n, both with torsion, whose geodesics correspond is left open. (Received November 20, 1940.)

71. Richard Courant: *Critical points and unstable minimal surfaces.*

Morse and Tompkins, and independently Shiffman (Annals of Mathematics, (2), vol. 40 (1939), pp. 834–854), have shown that Morse's theory of critical points in function spaces can be applied to minimal surfaces spanning suitably smooth contours. The main difficulty to overcome is the proof of a deformation property of the Dirichlet functional in the space of harmonic vectors. For this purpose a thorough analysis of the explicit Douglas boundary functional or its equivalent is needed. The present note, in line with the author's previous work on the Douglas problem, attacks the question for polygonal contours in a wider space by intrinsic considerations. It solves the problem by reducing it to that of the stationary points of a function of a finite number of variables with continuous derivatives. (Received November 27, 1940.)

72. N. A. Court: *On the harmonic pole.*

If U, V are two tetrahedrons (triangles) polar reciprocal for a quadric (conic), and L is the pole of a plane λ (line l) for the quadric (conic), the harmonic plane (line) of L for U and the harmonic pole of the plane λ (line l) for V are pole and polar plane (line) with respect to the quadric (conic). Various consequences of this proposition are considered, of which the following may be noted: The harmonic plane (line) of a point of a quadric (conic) for a tetrahedron (triangle) inscribed in the quadric (conic), and the harmonic pole, for the tangential tetrahedron (triangle), of the tangent plane (line) to the quadric (conic) at the point considered are polar for the quadric (conic). (Received November 23, 1940.)

73. N. A. Court: *On the skew cubic.*

Given a tetrahedron T whose faces osculate a skew cubic C_3 the harmonic pole for T of a variable osculating plane of C_3 lies on a fixed line j. Conversely, given T and j, the harmonic plane for T of a variable point of j osculates a skew cubic. The transforms of j in the three skew harmonic homologies having for axes the three pairs of opposite edges of T are axes of the cubic, and so are the transforms of j in the four homologies, of constant -3, of which the vertices and the respectively opposite faces of T are the centers and planes of homology. Various properties connected with these axes of C_3 are considered. (Received November 23, 1940.)

74. S. B. Myers: *Complete Riemannian manifolds of positive mean curvature.*

The author has proved previously that if, on a complete n-dimensional Riemannian manifold M, the curvature at every point and with respect to every pair of directions is greater than a fixed positive constant, then M is closed (compact) and so is its universal covering manifold. In the present paper the same conclusions are drawn from the weaker hypothesis that the mean curvature of M at every point and with respect to every direction is greater than a fixed positive constant. In particular, a complete space of constant positive mean curvature is closed, and so is its universal covering manifold. Such spaces are important in the general theory of relativity. (Received November 25, 1940.)

Statistics and Probability

75. G. A. Baker: *Fundamental distributions of errors for agricultural field trials.*

Evidence from various sources is presented which shows that the fundamental error distribution for yield trials is represented by $[1/ab(l_1-l_0)]\int_0^a\int_0^b\int_{l_0}^{l_1}[1/\sigma(x,y,t)]\exp-\frac{1}{2}\{[\xi-f(x,y,t)]^2/\sigma^2(x,y,t)\}\,d\xi dydx$ where the integrals may be Stieltjes integrals. Under certain conditions the fundamental error distribution can be expressed as a Gram-Charlier series, but very rarely, if ever, as a normal distribution. For comparison with analysis of variance results based on the normal theory, the distribution of the ratio of independent estimates of the second moments of samples, if the fundamental distributions are Gram-Charlier series, are given. Similar considerations show that the distributions of the numbers attacked in field trials can rarely be represented by Poisson or binomial distributions as is usually assumed. (Received October 22, 1940.)

76. G. A. Baker: *Maximum likelihood estimation of the ratio of the components of nonhomogeneous populations.*

Let $f(x)=[1/(1+k)](f_1(x)+kf_2(x))$, $e\leq x\leq f$, $k>0$ and $k<\infty$, where $f_1(x)$ and $f_2(x)$ are probability functions. The problem is to find the maximum likelihood estimate of k, say $\bar{k}$. If $f_1(x)$ and $f_2(x)$ are rectangular with equal ranges that partially overlap, then the probability of a value of $\bar{k}=w/u$ (where u is the number of individuals drawn from the nonoverlapped interval of $f_1(x)$, w is the number of individuals drawn from the nonoverlapped interval of $f_2(x)$ and v is the number of individuals drawn from the interval overlapped by $f_1(x)$ and $f_2(x)$) is $(n!/u!v!w!)(p_1)^u(p_2)^v(p_3)^w$ where the p_i's are the probabilities of coming from the respective intervals. The cases for which $u=n$, $v=n$, $w=n$, $u=0$, $w=0$ are excluded because $\bar{k}$ is then indeterminate. Hence, the probability of a determinate value of $\bar{k}$ is $P=1+(p_2)^n-(p_2+p_3)^n-(p_1+p_2)^n$. The estimates of k are biased. (Received October 22, 1940.)

77. G. F. McEwen: *Statistical problems of the range divided by the mean in samples of size n.*

Certain quantitative climatological studies are based upon the "precipitation ratio" or ratio to the mean annual rainfall of the difference between the maximum and minimum annual rainfall corresponding to the standard number of years. Available observations correspond to various values of the number of years n. Accordingly it is necessary to compute the precipitation ratio I corresponding to a standard number

of years from I_n corresponding to n years. Various assumptions regarding the frequency distribution of the precipitation lead to corresponding sets of theoretical factors for standardizing the precipitation ratio and certain of these agree well with observations. Results plotted on coordinate paper having a reciprocal scale for I_n and a logarithmic scale for n facilitate a graphical treatment of the observations since the graph approximates to a straight line. The standard error of I_n is also computed corresponding to the two most important frequency distributions of the precipitation. In order to reduce the accidental fluctuations of I_n it is sometimes desirable to use the difference between the averages of the rn highest and the rn lowest values, where rn equals some appropriate value, say 3. Theoretical factors derived for correcting such values to obtain I_n agree well with observations. (Received October 3, 1940.)

78. Jerzy Neyman: *A statistical problem in mass production and routine analyses.*

Let $x_{i1}, x_{i2}, \cdots, x_{in}$ denote the results of parallel analyses of some ith sample, $i=1, 2, \cdots, N$ (a sample of items of mass production). The x's are considered as random variables following the law $p(x_{ij})=(\sigma_i(2\pi)^{1/2})^{-1}\exp\{(x_{ij}-\xi_i)^2/2\sigma_i^2\}$. Denote by H the hypothesis that $\sigma_1=\sigma_2=\cdots=\sigma_N$, that is, that the precision of measuring ξ_i is maximum, and let $x_i=(\sum_j x_{ij})/n$, $S_i^2=\sum_j(x_{ij}-x_i)^2$, $V=\sum_i S_i^2$, $U=\sum_i S_i^4$. Also let E denote the observed point in the space W of the S_i^2's and $w(V)$ the part of the locus $V=$const. included in any region $w\subset W$. It is proved that (1) a necessary and sufficient condition for a test's probability of rejecting H when true to be a fixed value α is that its critical region $w=\sum_V w(V)$, with the $w(V)$ arbitrary except that $P\{E\in w(V)\mid E\in W(V)\}=\alpha$ for any $V>0$. (2) If H is not true and $h=\sigma^{-2}$ is a random variable with $p(h)=ch^{\alpha-1}e^{-\beta h}$, then the critical region w_0 determined by $U>U_\alpha V^2$, with $P\{U>U_\alpha V^2\mid H, V\}=\alpha$, has the following useful properties: (a) if H be true, then the probability of w_0 rejecting H is equal to α, (b) the first derivative of the power function of w_0, taken at the point of H being true, is equal to zero, (c) the second derivative is maximum. ((b) is true for all regions satisfying (a).) (Received October 28, 1940.)

Theory of Numbers

79. E. T. Bell: *Selective equations.*

The symbols $(\ )'$, $[\ ]'$ denote the least, greatest of the integers occurring within the symbols. Each such integer may be replaced by a symbol $(\ )'$, $[\ ]'$ referring to a new set of integers, and so on, a finite number of times. The most general system of equations consisting of equalities between single power products formed from $(\ )'$, $[\ ]'$ is solved non-tentatively and finitely, by passing to a unique dual of the system, in which $(\ )$, the G.C.D., and $[\ ]$, the L.C.M., replace $(\ )'$, $[\ ]'$. The latter system is a simple multiplicative system, and hence is completely solvable non-tentatively and finitely. The problem solved arose in the detailed discussion of compound multiplicative systems. (Received October 26, 1940.)

80. Leonard Carlitz: *Finite differences and polynomials in a Galois field.*

This paper is concerned chiefly with the solution of equations such as $\sum f(M)M(t)=g(t)$ and $\sum f(M, u)M(t)=g(u, t)$, the summation extending over all polynomials $M=M(x)$ of degree less than m. (Received November 25, 1940.)

81. Douglas Derry: *On the boundary case of Minkowski's linear form theorem.* Preliminary report.

Group theoretic methods are used to study Minkowski's conjecture regarding the boundary case of his linear form theorem, with the restriction that the coefficients become integral upon multiplication by a fixed prime p. In this case the validity of the conjecture is established for n forms provided $p>(n-3)/2$ and for all primes where $n\leqq 8$. (Received November 25, 1940.)

82. D. H. Lehmer: *A general summation formula for numerical functions.*

Let $h(n)$ be any single valued numerical function, and let $H(n)=h(1)+h(2)+\cdots+h(n)$ be its sum. The author obtains a formula for $H(n)$ in terms of two functions $f(n)$, $g(n)$ and their sum functions $F(n)$ and $G(n)$, such that $\sum f(\delta)g(n/\delta)=h(n)$, the sum extending over the divisors of n. The formula involves a parameter R which can be chosen at will. For $R=1$ or $R=n$ one obtains a formula used in several special cases by Dirichlet, while for $R=[n^{1/2}]$, a general formula of Franel and Glaisher is obtained, special cases of which have been used by many writers. By choosing R an appropriate function of n depending on h, the number of terms in the formula may be minimized. Exact formulas for the number of square-free numbers equal to or less than x and the excess of numbers equal to or less than x having an even number of prime factors over those having an odd number of prime factors are given as examples. In these formulas the number of terms varies as the cube root of x. (Received October 22, 1940.)

83. Gordon Pall: *Simultaneous representation in a quadratic and linear form.*

The problem of solving in integers x_i the pair of equations $a=\sum c_i x_i^2$, $b=\sum c_i x_i$, where the c_i, a, and b are given integers, and $t=\sum c_i\neq 0$, is reduced for certain forms of t to the representation of $ta-b^2$ in a specified form in one fewer variables. When this form is in a genus of one class the solution is complete. Twenty-nine such sets $(c_1, \cdots, c_4)$ with positive c_i are found. (Received November 25, 1940.)

84. R. M. Robinson: *On the simultaneous approximation of two real numbers.*

For each positive integer s, the smallest $C(s)$ is determined, such that for every pair of real numbers ξ_1 and ξ_2, there exist integers a_1, a_2, b with $0<b\leqq s$ and $|b\xi_k-a_k|\leqq C(s)$ $(k=1, 2)$. (Received September 28, 1940.)

85. J. B. Rosser: *Explicit bounds for some functions of prime numbers.*

In this paper the constants implicit in several "big O" theorems of the analytical theory of primes are evaluated and improved. A typical result is the following. Let $\pi(x)$ denote the number of primes less than or equal to x. Then for $55\leqq x$, $x/(\log x+2)<\pi(x)<x/(\log x-4)$. (Received November 23, 1940.)

86. H. A. Simmons: *Maximum numbers associated with a symmetric Diophantine equation in n reciprocals.*

In a previous paper (Duke Mathematical Journal, vol. 2 (1936), pp. 317–340) the

writer excluded from consideration the equation $1/x_1+1/x_2+\cdots+1/x_n+\lambda/x_1x_2\cdots x_n=b/((c+1)b-1)$, in which b, c are any positive integers, and λ is an integer greater than 1. The present paper identifies a class of maximum numbers relative to this equation. Using the terms E-solution and Kellogg solution, and the symbol $\Sigma_{n,r}(x)$ in senses previously employed, it is proved that if w is the Kellogg solution and if $X\neq w$ is any E-solution of the equation, then $\Sigma_{n,n-1}(w)>\Sigma_{n,n-1}(X)$ and $\Sigma_{n,n}(w)>\Sigma_{n,n}(X)$. On account of these two inequalities, it is possible to associate with this equation a class of infinitely many maximum numbers. It is also shown that $w_n>X_n$, which might be called the Kellogg property. (Received October 17, 1940.)

87. L. I. Wade: *Certain quantities transcendental over the field $GF(p^n, x)$.*

In a previous paper (abstract 46-1-129) the author proved certain quantities transcendental over the field $GF(p^n, x)$. The present paper includes certain additional transcendental quantities. For example, $\sum_{k=1}^{\infty}1/(x^{p^{nk}}-x)$ is transcendental, $\sum_{k=0}^{\infty}1/x^{q^k}$ $(q>1)$ is algebraic if q is of the form p^n and transcendental otherwise, and $\sum_{k=0}^{\infty}1/x^{k^q}$ $(q>1)$ is transcendental. (Received November 25, 1940.)

88. Max Zorn: *Idempotency of infinite cardinals.*

This paper contains a simple proof of the theorem that the product of two infinite numbers is equal to the maximal factor. The application of ordinal numbers is replaced by the use of the maximum principle. (Received October 28, 1940.)

Topology

89. Ben Dushnik and E. W. Miller: *On the dimension of a partial order.*

Let A be any set. Let K be any collection of linear orders, each defined on all of A. A partial order P on A is defined as follows: For any two elements a_1 and a_2 of A, $a_1<a_2$ (in P) if an only if $a_1<a_2$ in every linear order of the collection K. A partial order so obtained will be said to be realized by the linear orders of K. With the aid of a result due to Szpilrajn on the linear extensions of a partial order (Fundamenta Mathematicae, vol. 16 (1930), pp. 386–389), it is seen that if P is any partial order on a set A, then there exists a collection K of linear orders on A which realize P. By the dimension of a partial order P defined on a set A is meant the smallest cardinal number m such that P is realized by m linear orders on A. It is shown that if n is any natural number, there exists a finite partial order of dimension n, and if m is any transfinite cardinal, there exists a partial order of dimension m defined on a set of power m. Reversible partial orders (see abstract 46-5-266) are those of dimension not greater than 2. (Received October 25, 1940.)

90. Samuel Eilenberg: *Imbedding of spaces into euclidean spaces.* Preliminary report.

Given a metric space X, let $P(X)$ be the subset of the cartesian product $X\times X$ consisting of all points $(x, y)\in X\times X$ such that $x\neq y$. If every i-dimensional compact Vietoris cycle mod 2 of $P(X)$ bounds in $P(X)$ for $i=1, 2, \cdots, n-1$, then X is not imbeddable topologically into the euclidean n-space. Many other results of this type are obtained. Analogous theorems hold if the join $X\circ X$ is considered instead of the product $X\times X$. (Received November 25, 1940.)

91. Samuel Eilenberg: *Monotone families of manifolds.*

Given a closed n-manifold M in the euclidean space E^{n+1}, $D(M)$ will denote the bounded component of $E^{n+1}-M$. A family $\{M_t\}$ of n-manifolds depending on a real parameter t is called: (a) monotone if $D(M_{t_1}) \subset D(M_{t_2})$ for $t_1<t_2$, (b) strictly monotone if $M_{t_1} \subset D(M_{t_2})$ for $t_1<t_2$, (c) continuous if $\lim M_{t_n}=M_t$ whenever $\lim t_n=t$. If $\{M_t\}$ $(0<t\leqq 1)$ is monotone continuous and $\delta(M_t)\to 0$ as $t\to 0$, then $D(M_t)+M_t$ is contractible to a point and M_t is a homology-sphere. If moreover $\{M_t\}$ is strictly monotone, then also $D(M_t)$ is contractible to a point. (Received November 25, 1940.)

92. O. G. Harrold: *On a class of continuous maps.*

In this note the class of continua having the property of Čech (that is, of admitting a map of finite sections into the interval) is identified as those Peano continua M such that every dendrite D in M has the Čech property. For dendrites the problem was solved by Mazurkiewicz (Fundamenta Mathematicae, vol. 17, pp. 88–98). (Received November 23, 1940.)

93. F. B. Jones: *Topologically flat spaces.*

A nondegenerate continuous curve (in a complete Moore space) will be called topologically flat provided that (1) it contains no cut point, (2) the Jordan curve theorem holds true in it, and (3) it is locally remotely connected. Conditions are obtained under which a topologically flat continuous curve is a subset of a plane (complete separability being sufficient) and certain of these subsets are characterized by means of upper semicontinuous collections in the plane. (Received November 23, 1940.)

94. F. B. Jones: *Totally discontinuous linear functions whose graphs are connected.*

An example of a real function $f(x)$ of a real variable is constructed with the following properties: (1) $f(x+y)=f(x)+f(y)$; (2) $f(x)$ is defined for all real values of x and is discontinuous for each value of x; (3) the graph of $y=f(x)$ is a connected subset M of the number plane; (4) the set M is dense in the plane but contains no continuum; (5) considered as a space, (a) M is metric and linearly ordered, (b) M is the sum of a countable number of arbitrarily small totally disconnected domains, and (c) M contains a totally disconnected closed set which contains a domain. Other related problems are considered. (Received October 24, 1940.)

95. J. P. LaSalle: *Pseudo-normed linear sets over valued rings.* II.

A generalized concept of a linear function on a pseudo-normed linear set over a valued ring T (a P_1-space) to a P_1-space T' is introduced where the valued rings need not be the same, though a sort of generalized homogeneity is assumed. Bounded sets are defined for P_1-spaces, and it is shown that a linear function maps bounded sets into bounded sets. These generalized linear functions are shown to have others of the properties which linear functions possess in less general spaces. A differential with the usual properties can be defined for functions with arguments and values in P_1-spaces. When the P_1-spaces are in particular the "topological abelian groups" considered by Michal (*First order differentials of functions with arguments and values in topological abelian groups*, to be published in the Revista de Ciencas, University of Lima), the existence of the M_1-differential implies the existence of the differential considered in

this paper. By replacing the valued ring by a valued division ring in which the valuation satisfies $M(\alpha\beta)=M(\alpha)M(\beta)$ (it is not required that the valuation be archimedean), additional theorems concerning linear functions, and so on, may be proved. In particular it can be shown that the differential, if it exists, can be calculated as a limit. (Received November 8, 1940.)

96. A. N. Milgram: *Extensible and inextensible decompositions.* Preliminary report.

If A is a closed subset of a topological space S, and if A is decomposed into a finite number of closed sets $A=A_1+A_2+\cdots+A_r$, then we shall call the decomposition extensible if $S=S_1+S_2+\cdots+S_r$ is a decomposition of S into closed sets such that $S_i\supset A_i$ and the decomposition of S has the same nerve as that of A. If the decomposition of A is not extensible it will be called inextensible. (See Menger's *Dimensionstheorie*, pp. 312–313.) This paper is devoted to a study of extensible and inextensible decompositions of topological spaces. For example, it is shown that a compact space is of dimension not less than n if and only if it contains a closed subset having an inextensible decomposition into $n+1$ closed sets, each n of which intersect. Criteria to determine when a continuous mapping of a subset of a space can be extended to the whole space are given in terms of extensible decompositions. Further results will be reported in a later paper. (Received October 24, 1940.)

97. A. N. Milgram: *Extensions of decompositions.*

The system of sets $B_1, B_2, \cdots, B_s$ is said to govern the system $A_1, A_2, \cdots, A_s$, if $B_{i_1}\cdot B_{i_2}\cdot\cdots\cdot B_{i_k}=0$ implies $A_{i_1}\cdot A_{i_2}\cdot\cdots\cdot A_{i_k}=0$. For each absolute neighborhood retract R, there exists a decomposition into closed sets $R_1, R_2, \cdots, R_n$ which in a sense determines whether mappings of subspaces into R can be extended to mappings of the whole space into R. Namely, if $f\in R^A$, where A is a closed subset of the metric space S, then f can be extended to a mapping $f^*\in R^S$ if and only if there exist closed sets $S_i\supset f^{-1}(R_i)$, where $i=1, 2, \cdots, n$, which cover S and such that the system $S_1, S_2, \cdots, S_n$ is governed by the system $R_1, R_2, \cdots, R_n$. If R is not an absolute neighborhood retract, the theorem is shown to be false by an example. A space S is called connected between n of its closed sets $A_1, A_2, \cdots, A_n$ (see Menger, *Dimensionstheorie*, pp. 312–313) if for each decomposition of S into closed sets $S_1, S_2, \cdots, S_n$ such that $S_i\supset A_i$ we have $\prod S_i\neq 0$. It is shown that a separable metric space is of dimension not less than n if and only if it is connected between some $n+1$ of its closed sets. (Received November 26, 1940.)

98. Harlan C. Miller: *Concerning certain types of end points of compact continua.*

H. M. Gehman made a study of various definitions of end points of bounded plane continua. The following definition was not included: The point P of a compact continum M is said to be a terminal point of M if every irreducible subcontinuum of M which contains P is irreducible from P to some point. In this paper it is shown that this definition is not equivalent to any of those considered by Gehman, except for the case where M is a continuous curve. Among others the following results are established: A compact nondegenerate continuum every subcontinuum of which is unicoherent and decomposable has two terminal points. A compact continuum every subcontinuum of which is unicoherent and decomposable is irreducible about the set consisting of all its terminal points. If M is a compact hereditarily decomposable con-

tinuum and K is a subset of the set of all its terminal points, then $M-K$ is connected. (Received November 25, 1940.)

99. Harlan C. Miller: *On the existence of a certain type of irreducible continuum.*

In this paper the following theorem is proved: if M is a closed and compact subset of a line L lying in a plane E, there exists a compact continuum N lying in E such that (1) one of the complementary domains of L has no point in common with N, (2) the common part of N and L is M, and (3) if x and y are points of different components of M then N is irreducible from x to y. (Received November 25, 1940.)

100. Harlan C. Miller and R. L. Swain: *Properties of two intersecting arcs.*

Let space satisfy Axioms 0–5 of R. L. Moore (American Mathematical Society Colloquium Publications, vol. 13). The following definition is due to R. L. Moore: The arc AB is said to cross the arc CF provided there exist intervals $A'B'$ and $C'F'$ of AB and CF, respectively, and a simple closed curve J containing $C'F'$ such that A' and B' belong to different complementary domains of J and such that the arc segment $C'F'$ contains the common part of J and $A'B'$. Some necessary and sufficient conditions are given for crossing, and the crossing relation is shown to be reciprocal. Two arcs may cross at a point or interval, may touch at a point or interval, or may cross in every neighborhood of a point or interval. Among other results obtained in this paper are (1) if two arcs cross, they cross in every neighborhood of some component of their common part, (2) if two arcs cross, but cross at no component of their common part, and K is the set of all components H of their common part such that they cross in every neighborhood of H, then K is a perfect set of point sets. (Received November 25, 1940.)

101. S. B. Myers: *Compact groups whose elements are of finite order.*

Denote by G a metrizable topological group all of whose elements are of finite order, and by G_N a G which satisfies the postulate that if a sequence of elements g_i converges to the identity, then the order of g_i becomes infinite. (This postulate is satisfied by any G which is a group of periodic homeomorphisms of a manifold into itself.) In this note it is proved that a compact G_N is 0-dimensional, and a compact abelian G_N is finite. (Received November 25, 1940.)

102. J. W. Odle: *Non-separating and non-alternating transformations modulo a family of sets.*

In this paper the type of generalization of non-alternating and non-separating transformations introduced by E. P. Vance (Duke Mathematical Journal, vol. 6 (1940), pp. 66–79) is carried still further, and the most general possible transformations of this kind are shown to have the same characteristic properties as Vance's transformations. A systematic analysis of product and factor theorems involving monotone, non-separating, non-alternating, weakly non-separating, and weakly non-alternating transformations is made, and with the addition of the two new theorems in this paper, all possible theorems of this type are now known. The relationships between locally non-alternating and 0-regular transformations are studied, and it is proved that any 0-regular transformation is locally non-alternating. An example is

given to show that the converse is not true in general. However, on certain special sets the transformations are equivalent. (Received October 17, 1940.)

103. N. E. Rutt: *Sets with a certain sort of derived set.*

Let D be a set of objects. Suppose that corresponding to each subset K_a of D there is a second subset K_a' of D such that, when K_a' is nonvacuous and at most a finite number of K_a are not in K_b, then the product of K_a' and K_b' is nonvacuous. The sets K_a' exhibit a few of the simpler aspects of derived sets and the nature of these is the subject of this paper. (Received November 18, 1940.)

104. G. E. Schweigert: *Border transformations.*

The nature of these transformations may be inferred from the fact that for $B=T(A)$ a perfect set, where A is compact and metric, the transformation is interior if and only if T is a (continuous) border transformation such that the inverse image of a point contains no open set. Questions as to what extent open or connected open sets are preserved are considered in detail. These considerations yield necessary and sufficient conditions that T be border. On locally connected continua, theorems holding for T interior are generalized and a factor theorem shows that a border T is quasi-monotone. Some results showing the difference between border and interior transformations are also obtained. (Received November 25, 1940.)

105. R. H. Sorgenfrey: *Concerning triodic continua.*

It is the purpose of this paper to present several definitions of the term triodic, first defined by R. L. Moore in Fundamenta Mathematicae, vol. 13 (1929), p. 262, and to establish the following theorem: If the compact continuum M contains three continua which have a point in common and such that no one of them is a subset of the sum of the other two, then M contains a continuum N, three points P, Q, and O not belonging to N, and continua H, K, and L irreducible from N to P, Q, and O, respectively, each two of the continua H, K, and L having only N in common. (Received November 23, 1940.)

106. R. L. Swain: *Distance axioms in Moore spaces.*

A Moore space satisfies Axiom 0 and Axiom 1 (condition (4) omitted) of R. L. Moore (American Mathematical Society Colloquium Publications, vol. 13); a weak Moore space satisfies Axiom 0 and Axiom 1′, where Axiom 1′ is Axiom 1 except that the condition $\bar{g}\subset (R-B)+A$ is replaced by $g\subset (R-B)+A$. In the following axioms, X, Y, X_n, Y_n are points of a space S: (A) $d(XY)=d(YX)$ is a real number; $d(XY)=0$ if $X=Y$; $d(XY)>0$ if $X\neq Y$; (B) if $\lim d(YY_n)=0$, then $\liminf d(XY_n)\leqq d(XY)$; (C) if $\lim d(XX_n)=0$ and $\lim d(YY_n)=0$, then $\lim d(X_nY_n)=0$; (D) given X and given a positive number ϵ, there exists a positive number $\delta_{p\epsilon}$ such that if $\lim d(YY_n)=0$ and $d(XY_n)<\delta_{p\epsilon}$, then $d(XY)<\epsilon$. The point X is said to be a limit point of a point set M if, for each n, there exists a point X_n of M such that $\lim d(XX_n)=0$. In order that S be a weak Moore space, it is necessary and sufficient that S satisfy Axioms A, B, C. For S to be a Moore space, it is necessary and sufficient that S satisfy Axioms A, B, C, D. (Received November 25, 1940.)

107. R. L. Swain: *Linear metric space.*

A metric space S is called linear if for each three points X', Y', Z' of S there exist three points X, Y, Z such that $X+Y+Z=X'+Y'+Z'$ and $d(XZ)=d(XY)+d(YZ)$, where d is a metric distance function for S. If S does not consist of just four points,

it is shown that there exists a reversibly continuous transformation which preserves order and distance and which maps S into some subset of the x-axis. (Received November 25, 1940.)

108. P. M. Swingle: *An abstraction of types of connected sets.*

Let U and V be subsets of a set M such that $M=U+V$ where not both U and V are M. Let $S(X)$ denote that set X has property S, $U\ r\ V$ denote that U has relation r to V. A set M is of type (I, B, W) in P and r if and only if M has property P and, for every pair U and V with property P, $U\ r\ V$. Let M be a set of type (I, B, W). Then M is an SPQ-set of type B if and only if $U\ r\ V \equiv [S(U)\cdot S(V)$ has property $Q]$, M is an SPQ-set of type I if and only if $U\ r\ V \equiv$ [either $S(U)$ or $S(V)$ has property Q], and M is an SPQ-set of type W if and only if $U\ r\ V \equiv$ [both $S(U)$ and $S(V)$ have property Q]. Certain general theorems are obtained concerning these sets and particular theorems obtained assigning various properties to S, P and Q. One can take these such that types B, W and I are respectively biconnected, widely connected and indecomposable closed connected sets. (Received November 18, 1940.)

109. P. M. Swingle: *Indecomposable connexes.*

Let M be a connected point set. Then M is an indecomposable connex if and only if for every two connected subsets H, K of M such that $M=H+K$ either both H and M or both K and M have the same closure. And M is an irreducible joining connex closure between a and b if and only if there exists a connected subset N of M such that $N+a+b$ is connected and for all such N's the sets N and M have the same closure. Examples are given of these sets and among others the following theorem is proven: If M is connected and the closure of M is compact, then in order that M be an indecomposable connex it is necessary and sufficient that there exist three points x, y, z such that M is an irreducible joining connex closure between each two of these points. (Received October 26, 1940.)

110. A. W. Tucker: *Barycentric mappings.*

To each cell x of an augmentable complex X let there correspond a vertex $v(x)$ of a simplicial complex Y in such a way that $v(x), v(x'), \cdots, v(x^{(k)})$ are all vertices of some single simplex of Y if $x<x'<\cdots<x^{(k)}$ ($<$ means "is a face of"). Then $Tx=v(x)TFx$ defines a chain mapping of X into Y satisfying the usual requirement that $FTx=TFx$. And if Y is a closed subcomplex of X such that each x is joined in X to the simplex formed from the vertices $v(x), v(x'), \cdots, v(x^{(k)})$ where $x<x'<\cdots<x^{(k)}$, then $Dx=v(x)\{x-DFx\}$ defines a chain deformation such that $DFx+FDx=x-Tx$. The correspondence $x\to v(x)$ may be thought of as a simplicial mapping of the barycentric subdivision of X into Y—hence the name barycentric mapping. It has many convenient properties and a wide variety of uses, particularly in comparing one homology system with another. (Received October 26, 1940.)

111. C. W. Vickery: *A new proof of a theorem of Chittenden.*

It is shown that if S is a space $(\mathcal{E})$ in which the écart is regular (terminology of Fréchet's *Les Espaces Abstraits*, p. 219), there exists an equicontinuous collection G of real valued point functions defined over S and satisfying the conditions of Theorem 1.6 of the author's paper *Spaces of uncountably many dimensions* (this Bulletin, vol. 45 (1939), p. 459). Thus S is homeomorphic with a subset of a space D^{α} and hence is metric. This establishes a theorem of Chittenden (Transactions of this Society, vol. 18 (1917), p. 161). (Received October 12, 1940.)

112. R. L. Wilder: *Characterizations of those euclidean domains whose boundaries are lc^k.*

The central problem is to find positional properties which characterize those domains in the euclidean n-sphere whose boundaries are lc^k ($k \geqq 0$). The only solution previously given was for the case where $n=2$ and $k=0$ (R. L. Moore, Fundamenta Mathematicae, vol. 3 (1922), pp. 232–237). A complete solution is given in this paper, using the properties S_i defined in an earlier paper (American Journal of Mathematics, vol. 61 (1939), pp. 823–832) and a weak property S_i (equals WS_i) defined exactly as was S_i except that carriers of chains need not be self-compact: In order that the boundary of a domain D in the euclidean n-sphere should be lc^k, it is necessary and sufficient that D have (1) property WS_0^k and (2) property S_{n-k-1}^{n-1} and property S_{n-k-2} relative to bounding cycles. For $n=2$, $k=0$, this is equivalent to the result of Moore cited above, since S_0 implies WS_0. Weak uniform local i-connectedness (equals i-*wulc*) is also investigated, and among other results it is found that in order for a common boundary of two domains in the euclidean n-sphere to be lc^k it is necessary and sufficient that the domains be $wulc_0^k$. (Received November 23, 1940.)

113. R. L. Wilder: *On the domains complementary to continua having certain avoidability properties.*

G. T. Whyburn showed (American Journal of Mathematics, vol. 61 (1939), pp. 733–749) that in the plane the boundary of a domain complementary to a compact semi-locally connected continuum is peanian. The present paper employs the property of almost local i-avoidability (equals i-*ala*; see American Journal of Mathematics, vol. 61 (1939), p. 832, footnote 18) which for $i=0=p^0$ is equivalent to semi-local connectedness. Among the results obtained are the following: Let M be a subcontinuum of S^n which is ala_0^{n-2}. Then (1) if only finitely many of the "small" i-cycles ($i \leqq n-2$) of a complementary domain D are linearly independent relative to homologies (*lirh*) in D, $F(D)$ is peanian; (2) if the complementary domains of M form a null sequence and only finitely many "small" i-cycles ($0<i \leqq n-2$) of S^n-M are *lirh* in S^n-M, then M is lc^{n-2}; (3) under condition (2) if M is i-avoidable for $i \leqq n-2$, all but a finite number of complementary domain boundaries are g.c. $(n-1)$-m.'s. If M is an $(n-2)$-*ala* Peano continuum in S^n, then the complementary domain boundaries are all peanian (since every Peano continuum is 0-*ala*, this is exactly the well known Torhorst theorem when $n=2$). (Received November 23, 1940.)

114. R. L. Wilder: *The duality between the S-properties of closed sets and their complements in the euclidean n-sphere.*

Let property B_i denote property S_i relative to bounding cycles (American Journal of Mathematics, vol. 61 (1939), pp. 823–832). The following duality is established: In order that a closed subset M of the euclidean n-sphere S^n should have property B_i, it is necessary and sufficient that S^n-M have property B_{n-i-2}. Thus property S_i of S^n-M and finiteness of $p^{i+1}(S^n-M)$ are dual to property S_{n-i-2} of M and finiteness of $p^{n-i-1}(M)$, since for any set K to have property S_i is equivalent to having property B_i and finite $p^i(K)$. The duality is a consequence of the following results: (1) If M is a closed subset of S^n, then a necessary and sufficient condition that S^n-M have property B_i is that the closed subsets of M be almost completely $(n-i-2)$-avoidable by bounding cycles of M; and (2) a necessary and sufficient condition for a compact metric space to have property B_i is that the closed subsets of M be almost completely i-avoidable by bounding cycles. (Received November 23, 1940.)

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